

# Maximum Principle for Quasi-linear Reflected Backward SPDEs

Guanxing Fu<sup>1</sup>

Ulrich Horst<sup>1</sup>

Jinniao Qiu<sup>2</sup>

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## Abstract

This paper establishes a maximum principle for quasi-linear reflected backward stochastic partial differential equations (RBSPDEs for short). We prove the existence and uniqueness of the weak solution to RBSPDEs allowing for non-zero Dirichlet boundary conditions and, using a stochastic version of De Giorgi's iteration, establish the maximum principle for RBSPDEs on a general domain. The maximum principle for RBSPDEs on a bounded domain and the maximum principle for backward stochastic partial differential equations (BSPDEs for short) on a general domain can be obtained as byproducts. Finally, the local behavior of the weak solutions is considered.

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**Keywords:** reflected backward stochastic partial differential equation, backward stochastic partial differential equation, maximum principle, De Giorgi's iteration

## 1 Introduction

Let  $(\Omega, \mathcal{F}, \mathbb{P})$  be a complete probability space carrying a standard  $m$ -dimensional Brownian motion  $W = \{W_t, t \geq 0\}$ . Let  $(\mathcal{F}_t)_{t \geq 0}$  be the natural filtration generated by  $W$ , augmented by the  $\mathbb{P}$ -null sets in  $\mathcal{F}$ . In this paper, we establish a maximum principle for weak solutions to the reflected backward stochastic partial differential equation (RBSPDE)

$$\left\{ \begin{array}{l} -du(t, x) = [\partial_j(a^{ij}\partial_i u(t, x) + \sigma^{jr}v^r(t, x)) + f(t, x, u(t, x), \nabla u(t, x), v(t, x)) \\ \quad + \nabla \cdot g(t, x, u(t, x), \nabla u(t, x), v(t, x))] dt + \mu(dt, x) - v^r(t, x)dW_t^r, \\ \quad (t, x) \in Q := [0, T] \times \mathcal{O}, \\ u(T, x) = G(x), \quad x \in \mathcal{O}, \\ u(t, x) \geq \xi(t, x) \quad dt \times dx \times d\mathbb{P} - a.e., \\ \int_Q (u(t, x) - \xi(t, x))\mu(dt, dx) = 0, \end{array} \right. \quad (1.1)$$

with general Dirichlet boundary conditions. Here and in what follows, the usual summation convention is applied,  $\xi$  is a given stochastic process defined on  $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$ , called the *obstacle process*,  $T \in (0, \infty)$  is a deterministic *terminal time*,  $\mathcal{O} \subset \mathbb{R}^n$  is a possibly unbounded domain,  $\partial_j u = \frac{\partial u}{\partial x_j}$  and

<sup>1</sup>Department of Mathematics, Humboldt-Universität zu Berlin, Unter den Linden 6, 10099 Berlin, Germany. Financial support from the Berlin Mathematical School and the SFB 649 "Economic Risk" is gratefully acknowledged. *E-mail:* fuguangxing725@gmail.com (Guanxing Fu); horst@math.hu-berlin.de (Ulrich Horst).

<sup>2</sup>Department of Mathematics, University of Michigan, East Hall, 530 Church Street, Ann Arbor, MI 48109-1043, USA. *Email:* qiujiann@gmail.com (Jinniao Qiu).

$\nabla = (\partial_1, \dots, \partial_d)$  denotes the gradient operator. A *solution* to the RBSPDE is a random triple  $(u, v, \mu)$  defined on  $\Omega \times [0, T] \times \mathbb{R}^n$  such that (1.1) holds in a suitable sense.

Since the introduction by Bensoussan [2] backward stochastic partial differential equations (BSPDEs) have been extensively investigated in the probability and stochastic control literature. They naturally arise in many applications, for instance as stochastic Hamilton-Jacobi-Bellman equations associated with non-Markovian control problems [11], as adjoint equations of the Duncan-Mortensen-Zakai equation in nonlinear filtering [17] and as adjoint equations in stochastic control problems when formulating stochastic maximum principles [2]. BSPDEs with singular terminal conditions arise in non-Markovian models for financial mathematics to describe optimal trading in illiquid financial markets [9].

Reflected BSPDEs arise as the Hamilton-Jacobi-Bellman equation for the optimal stopping problem of stochastic differential equations with random coefficients [3, 16], and as the adjoint equations for the maximum principle of Pontryagin type in singular control problems of stochastic partial differential equations in, e.g. [10]

Existence and uniqueness of solutions results for reflected PDEs and SPDEs have been established by many authors. Pierre [12, 13] has studied parabolic PDEs with obstacles using parabolic potentials. Using methods and techniques from parabolic potential theory Denis, Matoussi and Zhang [7] proved existence and uniqueness of solutions results for quasi-linear SPDEs driven by infinite dimensional Brownian motion. More recently, Qiu and Wei [16] established a general theory of existence and uniqueness of solution for a class of quasi-linear RBSPDEs, which includes the classical results on obstacle problems for deterministic parabolic PDEs as special cases.

Adapting Moser's iteration scheme to the nonlinear case Aronson and Serrin [1] proved the maximum principle and local bounds of weak solutions for deterministic quasi-linear parabolic equations on bounded domains. Their method was extended by Denis, Matoussi, and Stoica [5] to the stochastic case, obtaining an  $L^p$  a priori estimate for the uniform norm of the solution of the stochastic quasi-linear parabolic equation with null Dirichlet condition, and further adapted by Denis, Matoussi, and Stoica [6] to local solutions. Later, Denis, Matoussi, and Zhang [8] established  $L^p$  estimates for the uniform norm in time and space of weak solutions to reflected quasi-linear SPDEs along with a maximum principle for local solutions using a stochastic version of Moser's iteration scheme. Recently, Qiu and Tang [15] used the De Giorgi's iteration scheme, a technique that also works for degenerate parabolic equations, to establish a local and global maximum principle for weak solutions of BSPDEs without reflection. To the best of our knowledge a maximum principle for reflected BSPDEs has not yet been established in the literature.

In this paper we establish a maximum principle for reflected BSPDEs on possibly unbounded domains; a maximum principle and a comparison principle for BSPDEs on general domains, a maximum principle for RBSPDEs on bounded domains and a local maximum principle for RBSPDEs are obtained as well. Due to the obstacle, the maximum principle for RBSPDE is not a direct extension of that for BSPDE in [15]. Our proofs rely on a stochastic version of De Giorgi's iteration scheme that does not depend on the Lebesgue measure of the domain; this extends the scheme in [15] that only applies to bounded domains. Our iteration scheme requires an almost sure representation of the  $L^2$  norm of the positive part of the weak solution of RBSPDEs. This, in turn requires generalizing the Itô's formula for weak solutions to BSPDEs established in [15] and [16] to the positive part of weak solutions.

It is worth pointing out that by contrast to  $L^p$  estimates ( $p \in (2, \infty)$ ) for the time and space maximal norm of weak solutions to *forward* SPDEs or related obstacle problems as established in [5, 8, 14], our estimate for weak solutions is uniform with respect to  $w \in \Omega$  and hence establishes an  $L^\infty$  estimate. This distinction comes from the essential difference between BSPDEs and *forward* SPDEs: the noise term in the former endogenously originates from martingale representation and is hence governed by the coefficients, while the latter is fully exogenous, which prevents any  $L^\infty$  estimate for *forward* SPDEs.

The paper is organized as follows: in Section 2, we list some notations and the standing assumptions on the parameters of the RBSPDE (1.1). The existence and uniqueness of weak solution to the RBSPDE (1.1) is presented in Section 3. In Sections 4, we establish the maximum principle for the RBSPDE (1.1) on a general domain as well as the maximum principles for RBSPDEs on a bounded domain and BSPDEs on a general domain. The local behavior of the weak solutions to (1.1) is also considered. Finally, we list in the appendix some useful lemmas, the frequently used Itô formulas and some definitions related to the stochastic regular measure.

## 2 Preliminaries and standing assumptions

For an arbitrary domain  $\Pi$  in some Euclidean space, let  $\mathcal{C}_0^\infty(\Pi)$  be the class of infinitely differentiable functions with compact support in  $\Pi$ , and  $L^2(\Pi)$  be the usual square integrable space on  $\Pi$  with the scalar product  $\langle u, v \rangle_\Pi = \int_\Pi u(x)v(x)dx$  and the norm  $\|u\|_{L^2(\Pi)} = \langle u, u \rangle_\Pi^{\frac{1}{2}}$  for each pair  $u, v \in L^2(\Pi)$ . For  $(k, p) \in \mathbb{Z} \times [1, \infty)$  where  $\mathbb{Z}$  is the set of all the integers, let  $H^{k,p}(\Pi)$  be the usual  $k$ -th order Sobolev space. For convenience, when  $\Pi = \mathcal{O}$ , we write  $\langle \cdot, \cdot \rangle$  and  $\|\cdot\|$  for  $\langle \cdot, \cdot \rangle_{\mathcal{O}}$  and  $\|\cdot\|_{L^2(\mathcal{O})}$  respectively. We recall that  $Q = [0, T] \times \mathcal{O}$ .

For  $t \in [0, T]$  and  $\Pi \subseteq \mathbb{R}^n$ , we put  $\Pi_t := [t, T] \times \Pi$ . Denote by  $H_{\mathcal{F}}^{k,p}(\Pi_t)$  the class of  $H^{k,p}(\Pi)$ -valued predictable processes on  $[t, T]$  such that for each  $u \in H_{\mathcal{F}}^{k,p}(\Pi_t)$  we have that

$$\|u\|_{H_{\mathcal{F}}^{k,p}(\Pi_t)} := \left( E \left[ \int_t^T \|u(s, \cdot)\|_{H^{k,p}(\Pi)}^p ds \right] \right)^{1/p} < \infty.$$

Let  $\mathcal{M}^{k,p}(\Pi_t)$  be the subspace of  $H_{\mathcal{F}}^{k,p}(\Pi_t)$  such that

$$\|u\|_{k,p;\Pi_t} := \left( \text{esssup}_{\omega \in \Omega} \sup_{s \in [t, T]} E \left[ \int_s^T \|u(\omega, \tau, \cdot)\|_{H_{\mathcal{F}}^{k,p}(\Pi)}^p d\tau | \mathcal{F}_s \right] \right)^{1/p} < \infty$$

and  $\mathcal{L}^\infty(\Pi_t)$  be the subspace of  $H_{\mathcal{F}}^{0,p}(\Pi_t)$  such that

$$\|u\|_{\infty;\Pi_t} := \text{esssup}_{(\omega, s, x) \in \Omega \times \Pi_t} |u(\omega, s, x)| < \infty.$$

Denote by  $\mathcal{L}^{\infty,p}(\Pi_t)$  the subspace of  $H_{\mathcal{F}}^{0,p}(\Pi_t)$  such that

$$\|u\|_{\infty,p;\Pi_t} := \text{esssup}_{(\omega, s) \in \Omega \times [t, T]} \|u(\omega, s, \cdot)\|_{L_p(\Pi)} < \infty.$$

Let  $\mathcal{V}_2(\Pi_t)$  be the class of all  $u \in H_{\mathcal{F}}^{1,2}(\Pi_t)$  such that

$$\|u\|_{\mathcal{V}_2(\Pi_t)} := \left( \|u\|_{\infty,2;\Pi_t}^2 + \|\nabla u\|_{0,2;\Pi_t}^2 \right)^{1/2} < \infty$$

and let  $\mathcal{V}_{2,0}(\Pi_t)$  be the subspace of  $\mathcal{V}_2(\Pi_t)$  for which

$$\lim_{r \rightarrow 0} \|u(s+r, \cdot) - u(s, \cdot)\|_{L^2(\Pi)} = 0 \quad \text{for all } s, s+r \in [t, T], \quad \text{a.s.}$$

**Standing Assumption.** *We assume throughout that the coefficients and the obstacle process of the RBSPDE (1.1) satisfy the following conditions. Denote by  $\mathbb{F}$  the  $\sigma$ -algebra generated by all predictable sets on  $\Omega \times [0, T]$  associated with  $(\mathcal{F}_t)_{t \geq 0}$ .*

( $\mathcal{A}_1$ ) *The random functions*

$$g(\cdot, \cdot, \cdot, X, Y, Z) : \Omega \times [0, T] \times \mathcal{O} \rightarrow \mathbb{R}^n \quad \text{and} \quad f(\cdot, \cdot, \cdot, X, Y, Z) : \Omega \times [0, T] \times \mathcal{O} \rightarrow \mathbb{R}$$

are  $\mathbb{F} \otimes \mathcal{B}(\mathcal{O})$ -measurable for any  $(X, Y, Z) \in \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^m$  and there exist positive constants  $L, \kappa$  and  $\beta$  such that for each  $(X_i, Y_i, Z_i) \in \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^m$ ,  $i = 1, 2$ ,

$$|g(\cdot, \cdot, \cdot, X_1, Y_1, Z_1) - g(\cdot, \cdot, \cdot, X_2, Y_2, Z_2)| \leq L|X_1 - X_2| + \frac{\kappa}{2}|Y_1 - Y_2| + \sqrt{\beta}|Z_1 - Z_2|$$

and

$$|f(\cdot, \cdot, \cdot, X_1, Y_1, Z_1) - f(\cdot, \cdot, \cdot, X_2, Y_2, Z_2)| \leq L(|X_1 - X_2| + |Y_1 - Y_2| + |Z_1 - Z_2|).$$

(A<sub>2</sub>) The coefficients  $a$  and  $\sigma$  are  $\mathbb{F} \otimes \mathcal{B}(\mathcal{O})$ -measurable and there exist positive constants  $\varrho > 1$ ,  $\lambda$  and  $\Lambda$  such that for each  $\eta \in \mathbb{R}^n$  and  $(\omega, t, x) \in \Omega \times [0, T] \times \mathcal{O}$ ,

$$\begin{aligned} \lambda|\eta|^2 &\leq (2a^{ij}(\omega, t, x) - \varrho\sigma^{ir}\sigma^{jr}(\omega, t, x))\eta^i\eta^j \leq \Lambda|\eta|^2 \\ |a(\omega, t, x)| + |\sigma(\omega, t, x)| &\leq \Lambda, \end{aligned}$$

and

$$\lambda - \kappa - \varrho'\beta > 0 \text{ with } \varrho' := \frac{\varrho}{\varrho - 1}.$$

(A<sub>3</sub>) The terminal value satisfies  $G \in L^\infty(\Omega, \mathcal{F}_T, L^2(\mathcal{O})) \cap L^\infty(\Omega, \mathcal{O})$  and for some  $p > \max\{n + 2, 2 + 4/n\}$ , one has

$$\begin{aligned} g_0 &:= g(\cdot, \cdot, \cdot, 0, 0, 0) \in \mathcal{M}^{0,p}(Q) \cap \mathcal{M}^{0,2}(Q) \\ f_0 &:= f(\cdot, \cdot, \cdot, 0, 0, 0) \in \mathcal{M}^{0, \frac{p(n+2)}{p+n+2}}(Q) \cap \mathcal{M}^{0,2}(Q). \end{aligned}$$

(A<sub>4</sub>) The obstacle process  $\xi$  is almost surely quasi-continuous (see Appendix for the definition) on  $Q$  and there exists a process  $\hat{\xi}$  such that  $\xi \leq \hat{\xi}$   $ds \times dx \times d\mathbb{P}$ -a.e., where  $\hat{\xi} \in \mathcal{V}_{2,0}(Q)$  together with some  $\hat{v} \in \mathcal{M}^{0,2}(Q)$  is a solution to BSPDE

$$\begin{cases} -d\hat{\xi}(t, x) = [\partial_j(a^{ij}\partial_i\hat{\xi}(t, x) + \sigma^{jr}\hat{v}^r(t, x)) + \hat{f}(t, x) + \nabla \cdot \hat{g}(t, x)]dt - \hat{v}^r(t, x)dW_t^r, \\ (t, x) \in Q, \\ \hat{\xi}(T, x) = \hat{G}(x), \quad x \in \mathcal{O}, \end{cases} \quad (2.1)$$

with the random functions  $\hat{f}$ ,  $\hat{g}$  and  $\hat{G}$  satisfying

$$\hat{G} \in L^\infty(\Omega, \mathcal{F}_T, L^2(\mathcal{O})) \cap L^\infty(\Omega, \mathcal{O}), \quad \hat{f} \in \mathcal{M}^{0, \frac{p(n+2)}{p+n+2}}(Q) \cap \mathcal{M}^{0,2}(Q), \quad \hat{g} \in \mathcal{M}^{0,p}(Q) \cap \mathcal{M}^{0,2}(Q).$$

(A<sub>5</sub>) The function  $x \mapsto g(\cdot, \cdot, \cdot, x, 0, 0)$  is uniformly Lipschitz continuous in norm:

$$\begin{aligned} \|g(\cdot, \cdot, \cdot, X_1, 0, 0) - g(\cdot, \cdot, \cdot, X_2, 0, 0)\|_{0,p;Q} &\leq L|X_1 - X_2|; \\ \|g(\cdot, \cdot, \cdot, X_1, 0, 0) - g(\cdot, \cdot, \cdot, X_2, 0, 0)\|_{0,2;Q} &\leq L|X_1 - X_2|. \end{aligned}$$

*Remark 2.1.* While the assumptions (A<sub>1</sub> – A<sub>4</sub>) are standard for the existence and uniqueness of solution, the assumption A<sub>5</sub> is required for the iteration scheme for proof of the maximum principle in Theorem 4.1 below, which follows easily from (A<sub>1</sub>) when the domain is bounded.

For the index  $p$  specified in (A<sub>3</sub>) and  $t \in [0, T]$ , define the functional  $A_p$  and  $B_2$  as follows:

$$A_p(l, h; \mathcal{O}_t) := \|l\|_{0, \frac{p(n+2)}{p+n+2}; \mathcal{O}_t} + \|h\|_{0,p; \mathcal{O}_t}, \quad (l, h) \in \mathcal{M}^{0, \frac{p(n+2)}{p+n+2}}(\mathcal{O}_t) \times \mathcal{M}^{0,p}(\mathcal{O}_t)$$

and

$$B_2(l, h; \mathcal{O}_t) := \|l\|_{0,2; \mathcal{O}_t} + \|h\|_{0,2; \mathcal{O}_t}, \quad (l, h) \in \mathcal{M}^{0,2}(\mathcal{O}_t) \times \mathcal{M}^{0,2}(\mathcal{O}_t).$$

In Sections 3 and 4, we will repeatedly use the Young inequality of the form

$$\langle f, g \rangle = \langle \sqrt{\epsilon}f, \frac{1}{\sqrt{\epsilon}}g \rangle \leq \frac{1}{2} \left[ \epsilon \|f\|^2 + \frac{1}{\epsilon} \|g\|^2 \right]. \quad (2.2)$$

### 3 Existence and uniqueness of weak solution to RBSPDE (1.1)

In this section we prove an existence and uniqueness of weak solutions result for the RBSPDE (1.1) along with a strong norm estimate. The difficulty in defining weak solutions to the RBSPDE (1.1) is the random measure  $\mu$ . It is typically a local time so the Skorokhod condition  $\int_Q (u - \xi) \mu(dt, dx) = 0$  might not make sense. To give a rigorous meaning to the integral condition, the theory of parabolic potential and capacity introduced by [12, 13] was generalized by [16] to a backward stochastic framework. We recall the definition of quasi continuity and stochastic regular measures in Appendix B.

**Definition 3.1.** The triple  $(u, v, \mu)$  is called a weak solution to the RBSPDE (1.1) if:

- (1)  $(u, v) \in \mathcal{V}_{2,0}(Q) \times \mathcal{M}^{0,2}(Q)$  and  $\mu$  is a stochastic regular measure;
- (2) the RBSPDE (1.1) holds in the weak sense, i.e., for each  $\varphi \in \mathcal{C}_0^\infty(\mathbb{R}^+) \otimes \mathcal{C}_0^\infty(\mathcal{O})$ , we have

$$\begin{aligned} & \langle u(t, \cdot), \varphi(t, \cdot) \rangle \\ &= \langle G(\cdot), \varphi(T, \cdot) \rangle - \int_t^T \left\{ \langle u(s, \cdot), \partial_s \varphi(s, \cdot) \rangle + \langle \partial_j \varphi(s, \cdot), a^{ij}(s, \cdot) \partial_i u(s, \cdot) + \sigma^{jr} v^r(s, \cdot) \rangle \right\} ds \\ &+ \int_t^T \left[ \langle f(s, \cdot, u(s, \cdot), \nabla u(s, \cdot), v(s, \cdot)), \varphi(s, \cdot) \rangle - \langle g^j(s, \cdot, u(s, \cdot), \nabla u(s, \cdot), v(s, \cdot)), \partial_j \varphi(s, \cdot) \rangle \right] ds \\ &+ \int_{[t,T] \times \mathcal{O}} \varphi(s, x) \mu(ds, dx) - \int_t^T \langle \varphi(s, \cdot), v^r(s, \cdot) dW_s^r \rangle, \quad \text{a.s.;} \end{aligned}$$

- (3)  $u$  admits a quasi-continuous version  $\tilde{u}$  such that  $\tilde{u} \geq \xi$   $ds \times dx \times d\mathbb{P}$  a.e. and

$$\int_Q (\tilde{u}(t, x) - \xi(t, x)) \mu(dt, dx) = 0 \quad \mathbb{P}\text{-a.s.} \quad (3.1)$$

We denote by  $\mathcal{U}(\xi, f, g, G)$  the set of all the weak solutions of the RBSPDE (1.1) associated with the obstacle process  $\xi$ , the terminal condition  $G$ , and the coefficients  $f$  and  $g$ . Further,  $\mathcal{U}(-\infty, f, g, G)$  is the set of solutions when there is no obstacle, i.e.,  $\mathcal{U}(-\infty, f, g, G)$  is the set of solution pairs  $(u, v)$  to the associated BSPDE with terminal condition  $G$  and coefficients  $f$  and  $g$ .

The following theorem guarantees the existence and uniqueness of weak solutions in the sense of Definition 3.1. The arguments for the norm estimate also apply to Lemma 4.3 below, which is needed for the proof of our maximum principle.

**Theorem 3.2.** *Suppose that Assumptions  $(\mathcal{A}_1)$ – $(\mathcal{A}_4)$  hold and that  $\hat{\xi}|_{\partial\mathcal{O}} = 0$ . Then the RBSPDE (1.1) admits a unique solution  $(u, v, \mu)$  that satisfies the zero Dirichlet condition  $u|_{\partial\mathcal{O}} = 0$ . Moreover, for each  $t \in [0, T]$ , one has*

$$\begin{aligned} \|u\|_{\mathcal{V}_2(\mathcal{O}_t)} + \|v\|_{0,2;\mathcal{O}_t} &\leq C \left( \text{esssup}_{\omega \in \Omega} \|G(\omega, \cdot)\|_{L^2(\mathcal{O})} + \text{esssup}_{\omega \in \Omega} \|\hat{G}(\omega, \cdot)\|_{L^2(\mathcal{O})} \right. \\ &\quad \left. + B_2(f_0, g_0; \mathcal{O}_t) + B_2(\hat{f}, \hat{g}; \mathcal{O}_t) \right), \end{aligned} \quad (3.2)$$

where the positive constant  $C$  only depends on the constants  $\lambda, \varrho, \kappa, \beta, L$  and  $T$ .

*Proof.* It has been shown in [16, Theorem 4.12] that the RBSPDE (1.1) admits a unique solution  $(u, v, \mu)$  satisfying the zero Dirichlet condition  $u|_{\partial\mathcal{O}} = 0$  and that this solutions satisfies the integrability condition

$$E \left[ \sup_{t \in [0, T]} \|u(t)\|^2 \right] + E \left[ \int_0^T \|\nabla u(t)\|^2 dt \right] + E \left[ \int_0^T \|v(t)\|^2 dt \right] < \infty.$$

Hence, we only need to prove the estimate (3.2). To this end, notice first that

$$\begin{aligned}
& \int_t^T \int_{\mathcal{O}} (u(s, x) - \hat{\xi}(s, x)) \mu(ds dx) \\
&= \int_t^T \int_{\mathcal{O}} (u(s, x) - \xi(s, x) + \xi(s, x) - \hat{\xi}(s, x)) \mu(ds dx) \\
&\leq 0.
\end{aligned}$$

Thus for each  $t \in [0, T]$ , Proposition A.4 yields almost surely,

$$\begin{aligned}
& \|u(t) - \hat{\xi}(t)\|^2 + \int_t^T \|v(s) - \hat{v}(s)\|^2 ds \\
&= \|G - \hat{G}\|^2 - \int_t^T \langle u(s) - \hat{\xi}(s), v^r(s) - \hat{v}^r(s) \rangle dW_s^r \\
&\quad - \int_t^T \langle 2\partial_j(u - \hat{\xi}(s)), a^{ij}\partial_i(u - \hat{\xi})(s) + \sigma^{jr}(v^r - \hat{v}^r) \rangle ds \\
&\quad - \int_t^T \langle 2\partial_j(u - \hat{\xi}(s)), g^j(s, u(s), \nabla u(s), v(s)) - \hat{g}^j(s) \rangle ds \\
&\quad + \int_t^T \langle 2(u - \hat{\xi}(s)), f(s, u(s), \nabla u(s), v(s)) - \hat{f}(s) \rangle ds \\
&\quad + \int_{\mathcal{O}_t} 2(u(s, x) - \hat{\xi}(s, x)) \mu(ds, dx) \\
&\leq \|G - \hat{G}\|^2 - \int_t^T \langle u(s) - \hat{\xi}(s), v^r(s) - \hat{v}^r(s) \rangle dW_s^r \\
&\quad - \int_t^T \langle 2\partial_j(u - \hat{\xi}(s)), a^{ij}\partial_i(u - \hat{\xi})(s) + \sigma^{jr}(v^r - \hat{v}^r) \rangle ds \\
&\quad - \int_t^T \langle 2\partial_j(u - \hat{\xi}(s)), g^j(s, u(s), \nabla u(s), v(s)) - \hat{g}^j(s) \rangle ds \\
&\quad + \int_t^T \langle 2(u - \hat{\xi}(s)), f(s, u(s), \nabla u(s), v(s)) - \hat{f}(s) \rangle ds.
\end{aligned} \tag{3.3}$$

Applying assumption  $(\mathcal{A}_2)$  and (2.2), one has

$$\begin{aligned}
I_1 &:= -E \left[ \int_t^T \langle 2\partial_j(u - \hat{\xi}(s)), a^{ij}\partial_i(u - \hat{\xi})(s) + \sigma^{jr}(v^r - \hat{v}^r) \rangle ds \Big| \mathcal{F}_t \right] \\
&= -E \left[ \int_t^T \langle \partial_j(u - \hat{\xi}(s)), (2a^{ij} - \sigma^{ir}\sigma^{jr}\varrho)\partial_i(u - \hat{\xi})(s) + \sigma^{ir}\sigma^{jr}\varrho\partial_i(u - \hat{\xi})(s) + 2\sigma^{jr}(v^r - \hat{v}^r) \rangle ds \Big| \mathcal{F}_t \right] \\
&\leq -\lambda E \left[ \int_t^T \|\nabla(u(s) - \hat{\xi}(s))\|^2 ds \Big| \mathcal{F}_t \right] + \frac{1}{\varrho} E \left[ \int_t^T \|v(s) - \hat{v}(s)\|^2 ds \Big| \mathcal{F}_t \right].
\end{aligned}$$

By assumptions  $(\mathcal{A}_1)$  and  $(\mathcal{A}_3)$  and the estimate (2.2) it holds for each  $\epsilon > 0$  and  $\theta > 0$  that:

$$\begin{aligned}
I_2 &:= -E \left[ \int_t^T \langle 2\partial_j(u - \hat{\xi}(s)), g^j(s, u(s), \nabla u(s), v(s)) - \hat{g}^j(s) \rangle ds \Big| \mathcal{F}_t \right] \\
&\leq E \left[ \int_t^T \langle 2|\nabla(u(s) - \hat{\xi}(s))|, L|u(s)| + \frac{\kappa}{2}|\nabla u(s)| + \sqrt{\beta}|v(s)| \rangle ds \Big| \mathcal{F}_t \right] \\
&\quad + E \left[ \int_t^T \langle 2|\nabla(u(s) - \hat{\xi}(s))|, |\hat{g}(s)| + |g_0(s)| \rangle ds \Big| \mathcal{F}_t \right]
\end{aligned}$$

$$\begin{aligned}
&\leq 2\epsilon E \left[ \int_t^T \|\nabla(u(s) - \hat{\xi}(s))\|^2 ds | \mathcal{F}_t \right] + C(\epsilon) E \left[ \int_t^T \|g_0(s)\|^2 ds | \mathcal{F}_t \right] + C(\epsilon) E \left[ \int_t^T \|\hat{g}(s)\|^2 ds | \mathcal{F}_t \right] \\
&\quad + E \left[ \int_t^T \langle 2|\nabla(u(s) - \hat{\xi}(s))|, L|u(s) - \hat{\xi}(s)| + L|\hat{\xi}(s)| \rangle ds | \mathcal{F}_t \right] \\
&\quad + E \left[ \int_t^T \langle 2|\nabla(u(s) - \hat{\xi}(s))|, \frac{\kappa}{2}|\nabla(u(s) - \hat{\xi}(s))| + \frac{\kappa}{2}|\nabla\hat{\xi}(s)| \rangle ds | \mathcal{F}_t \right] \\
&\quad + E \left[ \int_t^T \langle 2|\nabla(u(s) - \hat{\xi}(s))|, \sqrt{\beta}|v(s) - \hat{v}(s)| + \sqrt{\beta}|\hat{v}(s)| \rangle ds | \mathcal{F}_t \right] \\
&\leq 2\epsilon E \left[ \int_t^T \|\nabla(u(s) - \hat{\xi}(s))\|^2 ds | \mathcal{F}_t \right] + C(\epsilon) E \left[ \int_t^T \|\hat{g}(s)\| ds | \mathcal{F}_t \right] + C(\epsilon) E \left[ \int_t^T \|g_0(s)\| ds | \mathcal{F}_t \right] \\
&\quad + 2\epsilon E \left[ \int_t^T \|\nabla(u(s) - \hat{\xi}(s))\|^2 ds | \mathcal{F}_t \right] + C(\epsilon, L) E \left[ \int_t^T \|u(s) - \hat{\xi}(s)\|^2 ds | \mathcal{F}_t \right] + C(\epsilon, L) E \left[ \int_t^T \|\hat{\xi}(s)\|^2 ds | \mathcal{F}_t \right] \\
&\quad + \kappa E \left[ \int_t^T \|\nabla(u(s) - \hat{\xi}(s))\|^2 ds | \mathcal{F}_t \right] + \epsilon E \left[ \int_t^T \|\nabla(u(s) - \hat{\xi}(s))\|^2 ds | \mathcal{F}_t \right] + C(\epsilon, \kappa) E \left[ \int_t^T \|\nabla\hat{\xi}(s)\|^2 ds | \mathcal{F}_t \right] \\
&\quad + \beta\theta E \left[ \int_t^T \|\nabla(u(s) - \hat{\xi}(s))\|^2 ds | \mathcal{F}_t \right] + \frac{1}{\theta} E \left[ \int_t^T \|v(s) - \hat{v}(s)\|^2 ds | \mathcal{F}_t \right] \\
&\quad + \epsilon E \left[ \int_t^T \|\nabla(u(s) - \hat{\xi}(s))\|^2 ds | \mathcal{F}_t \right] + C(\epsilon, \beta) E \left[ \int_t^T \|\hat{v}(s)\|^2 ds | \mathcal{F}_t \right] \\
&\leq (6\epsilon + \kappa + \beta\theta) E \left[ \int_t^T \|\nabla(u - \hat{\xi})(s)\|^2 ds | \mathcal{F}_t \right] + \frac{1}{\theta} E \left[ \int_t^T \|v(s) - \hat{v}(s)\|^2 ds | \mathcal{F}_t \right] \\
&\quad + C(\epsilon) E \left[ \int_t^T \|g_0(s)\|^2 ds | \mathcal{F}_t \right] + C(\epsilon) E \left[ \int_t^T \|\hat{g}(s)\|^2 ds | \mathcal{F}_t \right] \\
&\quad + C(\epsilon, L) E \left[ \int_t^T \|u(s) - \hat{\xi}(s)\|^2 ds | \mathcal{F}_t \right] + C(\epsilon, L) E \left[ \int_t^T \|\hat{\xi}(s)\|^2 ds | \mathcal{F}_t \right] \\
&\quad + C(\epsilon, \kappa) E \left[ \int_t^T \|\nabla\hat{\xi}(s)\|^2 ds | \mathcal{F}_t \right] + C(\epsilon, \beta) E \left[ \int_t^T \|\hat{v}(s)\|^2 ds | \mathcal{F}_t \right].
\end{aligned}$$

It follows from  $(\mathcal{A}_3)$  that:

$$\begin{aligned}
I_3 &:= E \left[ \int_t^T \langle 2(u(s) - \hat{\xi}(s)), f(s, u(s), \nabla u(s), v(s)) - \hat{f}(s) \rangle ds | \mathcal{F}_t \right] \\
&\leq E \left[ \int_t^T \langle 2|u(s) - \hat{\xi}(s)|, |f_0| + L|u(s)| + L|\nabla u(s)| + L|v(s)| + |\hat{f}(s)| \rangle ds | \mathcal{F}_t \right].
\end{aligned}$$

In view of (2.2) it further holds for each  $\epsilon_1 > 0$  that:

$$\begin{aligned}
&E \left[ \int_t^T \langle 2|u(s) - \hat{\xi}(s)|, L|u(s)| + L|\nabla u(s)| + L|v(s)| \rangle ds | \mathcal{F}_t \right] \\
&\leq \epsilon_1 E \left[ \int_t^T \|\nabla u(s)\|^2 ds | \mathcal{F}_t \right] + \epsilon_1 E \left[ \int_t^T \|v(s)\|^2 ds | \mathcal{F}_t \right] + C(\epsilon_1, L) E \left[ \int_t^T \|u(s) - \hat{\xi}(s)\|^2 ds | \mathcal{F}_t \right] \\
&\quad + E \left[ \int_t^T \langle 2|(u - \hat{\xi})(s)|, L|(u - \hat{\xi})(s)| + L|\hat{\xi}(s)| \rangle ds | \mathcal{F}_t \right]
\end{aligned}$$

$$\begin{aligned}
&\leq 2\epsilon_1 E \left[ \int_t^T \|\nabla(u(s) - \hat{\xi}(s))\|^2 ds | \mathcal{F}_t \right] + 2\epsilon_1 E \left[ \int_t^T \|v(s) - \hat{v}(s)\|^2 ds | \mathcal{F}_t \right] + 2\epsilon_1 E \left[ \int_t^T \|\hat{v}(s)\|^2 ds | \mathcal{F}_t \right] \\
&\quad + 2\epsilon_1 E \left[ \int_t^T \|\nabla \hat{\xi}(s)\|^2 ds | \mathcal{F}_t \right] + C(\epsilon_1, L) E \left[ \int_t^T \|u(s) - \hat{\xi}(s)\|^2 ds | \mathcal{F}_t \right] + E \left[ \int_t^T \|\hat{\xi}(s)\|^2 ds | \mathcal{F}_t \right],
\end{aligned}$$

and by the Hölder inequality one has that:

$$\begin{aligned}
&E \left[ \int_t^T \langle 2|u(s) - \hat{\xi}(s)|, |f_0| + |\hat{f}(s)| \rangle ds | \mathcal{F}_t \right] \\
&\leq 2E \left[ \int_t^T \|(u - \hat{\xi})(s)\|^2 ds | \mathcal{F}_t \right] + E \left[ \int_t^T \|f_0(s)\|^2 ds | \mathcal{F}_t \right] + E \left[ \int_t^T \|\hat{f}(s)\|^2 ds | \mathcal{F}_t \right]
\end{aligned}$$

In addition,

$$I_4 := E \left[ \|G - \hat{G}\|^2 | \mathcal{F}_t \right] \leq \text{esssup}_{\omega \in \Omega} \|G - \hat{G}\|^2.$$

Summing up the estimates  $I_1$ - $I_4$  and taking the supremum w.r.t.  $(\omega, s) \in \Omega \times [t, T]$  on both sides we arrive at:

$$\begin{aligned}
&\|u - \hat{\xi}\|_{\infty, 2; \mathcal{O}_t}^2 + \|v - \hat{v}\|_{0, 2; \mathcal{O}_t}^2 + (\lambda - \kappa - \beta\theta - 6\epsilon - 2\epsilon_1) \|\nabla(u - \hat{\xi})\|_{0, 2; \mathcal{O}_t}^2 \\
&\leq \left( \frac{1}{\varrho} + \frac{1}{\theta} + 2\epsilon_1 \right) \|v - \hat{v}\|_{0, 2; \mathcal{O}_t}^2 + C(\epsilon, \epsilon_1, L) \int_t^T \|u - \hat{\xi}\|_{\infty, 2; \mathcal{O}_s}^2 ds + C(\epsilon, \epsilon_1, \beta) \|\hat{v}\|_{0, 2; \mathcal{O}_t}^2 \\
&\quad + C(\epsilon, \epsilon_1, \kappa, L) \|\hat{\xi}\|_{\mathcal{V}_2(\mathcal{O}_t)}^2 + \|f_0\|_{0, 2; \mathcal{O}_t}^2 + \|\hat{f}\|_{0, 2; \mathcal{O}_t}^2 + C(\epsilon) (\|g_0\|_{0, 2; \mathcal{O}_t}^2 + \|\hat{g}\|_{0, 2; \mathcal{O}_t}^2) \\
&\quad + \text{esssup}_{\omega \in \Omega} \|G - \hat{G}\|^2.
\end{aligned}$$

By assumption  $(\mathcal{A}_2)$  we can choose  $\theta > \varrho'$  such that  $\lambda - \kappa - \beta\theta > 0$ , and  $\theta > \varrho'$  also implies  $\frac{1}{\varrho} + \frac{1}{\theta} < 1$ . Now taking  $\epsilon$  and  $\epsilon_1$  small enough such that  $\lambda - \kappa - \beta\theta - 6\epsilon - 2\epsilon_1 > 0$  and  $\frac{1}{\varrho} + \frac{1}{\theta} + 2\epsilon_1 < 1$ , we have

$$\begin{aligned}
&\|u - \hat{\xi}\|_{\mathcal{V}_2(\mathcal{O}_t)}^2 + \|v - \hat{v}\|_{0, 2; \mathcal{O}_t}^2 \\
&\leq C(\epsilon, \epsilon_1, \lambda, \beta, \kappa, L, \varrho) \left( \int_t^T \|u - \hat{\xi}\|_{\infty, 2; \mathcal{O}_s}^2 ds + \|\hat{v}\|_{0, 2; \mathcal{O}_t}^2 + \|\hat{\xi}\|_{\mathcal{V}_2(\mathcal{O}_t)}^2 \right. \\
&\quad \left. + B_2(f_0, g_0; \mathcal{O}_t)^2 + B_2(\hat{f}, \hat{g}; \mathcal{O}_t)^2 + \text{esssup}_{\omega \in \Omega} \|G - \hat{G}\|_{L^2(\mathcal{O})}^2 \right).
\end{aligned} \tag{3.4}$$

By Gronwall's inequality,

$$\begin{aligned}
&\|u - \hat{\xi}\|_{\mathcal{V}_2(\mathcal{O}_t)}^2 + \|v - \hat{v}\|_{0, 2; \mathcal{O}_t}^2 \\
&\leq C(\epsilon, \epsilon_1, \lambda, \beta, \kappa, L, \varrho, T) \left( \|\hat{v}\|_{0, 2; \mathcal{O}_t}^2 + \|\hat{\xi}\|_{\mathcal{V}_2(\mathcal{O}_t)}^2 + \text{esssup}_{\omega \in \Omega} \|G - \hat{G}\|_{L^2(\mathcal{O}_t)}^2 \right. \\
&\quad \left. + B_2(f_0, g_0; \mathcal{O}_t)^2 + B_2(\hat{f}, \hat{g}; \mathcal{O}_t)^2 \right).
\end{aligned} \tag{3.5}$$

Since  $\hat{\xi}|_{\partial\mathcal{O}} = 0$ , we can apply Proposition A.4 to  $\|\hat{\xi}(t)\|^2$ . Starting from (3.3), using similar estimates,

$$\|\hat{\xi}\|_{\mathcal{V}_2(\mathcal{O}_t)}^2 + \|\hat{v}\|_{0, 2; \mathcal{O}_t}^2 \leq C \left( B_2(\hat{f}, \hat{g}; \mathcal{O}_t)^2 + \text{esssup}_{\omega \in \Omega} \|\hat{G}\|^2 \right), \tag{3.6}$$

where  $C$  only depends on  $\lambda, \beta, \kappa, \varrho, L$  and  $T$ . The estimate (3.5) together with (3.6) yields (3.2).  $\square$

With the same notation as in Theorem 3.2, we can relax the zero Dirichlet boundary condition in Theorem 3.2 by assuming  $u|_{\partial\mathcal{O}} = \tilde{u}|_{\partial\mathcal{O}}$  for some  $(\tilde{u}, \tilde{v}) \in \mathcal{U}(-\infty, \tilde{f}, \tilde{g}, \tilde{G})$  where the coefficients  $a, \sigma, \tilde{f}, \tilde{g}$  and  $\tilde{G}$  satisfy  $(\mathcal{A}_2)$  and  $(\mathcal{A}_3)$  respectively, and  $\tilde{f}$  and  $\tilde{g}$  do not depend on  $\tilde{u}, \nabla\tilde{u}$  and  $\tilde{v}$ . Assume further that



$\xi|_{\partial\mathcal{O}} \leq \tilde{u}|_{\partial\mathcal{O}}$  and put  $\bar{\xi} := \hat{\xi} - \tilde{u}$ . Then,  $(\bar{\xi}, \bar{v}) \in \mathcal{U}(-\infty, \bar{f}, \bar{g}, \bar{G})$ , where  $\bar{v} = \hat{v} - \tilde{v}$ ,  $\bar{f} = \hat{f} - \tilde{f}$ ,  $\bar{g} = \hat{g} - \tilde{g}$  and  $\bar{G} = \hat{G} - \tilde{G}$ . Suppose now that  $(\check{\xi}, \check{v}) \in \mathcal{U}(-\infty, \check{f}, \check{g}, \check{G})$  with  $\check{\xi}|_{\partial\mathcal{O}} = 0$ . Then,  $\check{\xi}|_{\partial\mathcal{O}} = 0 \geq (\hat{\xi} - \tilde{u})|_{\partial\mathcal{O}} = \bar{\xi}|_{\partial\mathcal{O}}$  and the maximum principle in Lemma 4.4 yields  $\check{\xi} \geq \hat{\xi} - \tilde{u} \geq \bar{\xi}$ . Therefore, our RBSPDE (1.1) is equivalent to the following one but with zero-Dirichlet condition:

$$\left\{ \begin{array}{l} -d\check{u}(t, x) = [\partial_j(a^{ij}\partial_i\check{u} + \sigma^{jr}\check{v}^r)(t, x) + (f + \nabla \cdot g)(t, x, \check{u} + \tilde{u}, \nabla(\check{u} + \tilde{u}), \check{v} + \tilde{v}) \\ \quad - (\tilde{f} + \nabla \cdot \tilde{g})(t, x)] dt + \mu(dt, x) - \check{v}^r(t, x)dW_t^r, \quad (t, x) \in Q; \\ \check{u}(T, x) = G(x) - \tilde{G}(x), \quad x \in \mathcal{O}; \\ \check{u} \geq \xi - \tilde{u}, \quad \mathbb{P} \otimes dt \otimes dx\text{-a.e.}; \\ \int_Q (\check{u} - (\xi - \tilde{u}))(t, x) \mu(dt, dx) = 0. \end{array} \right. \quad (3.7)$$

By Theorem 3.2, there is a unique solution  $u - \tilde{u}$  to the RBSPDE (3.7) satisfying zero-Dirichlet condition. In this way, Theorem 3.2 extends to RBSPDEs with general Dirichlet conditions.

## 4 Maximum Principle for RBSPDE

In this section we state and prove our maximum principles for RBSPDEs. We start with a global maximum principle on general domains, which states that the weak solution  $u$  is bounded on the whole domain if it is bounded on the parabolic boundary. Subsequently we analyze the local behavior of  $u^\pm$  when  $u$  is not necessarily bounded on the parabolic boundary.

### 4.1 Global Case

This section establishes a maximum principle for the RBSPDE (1.1) on a general domain  $\mathcal{O}$ . Since the Lebesgue measure of  $\mathcal{O}$  might not be bounded, the scheme in [15] cannot be applied. Instead, motivated by [14], we use a stochastic De Giorgi's scheme that is independent of the measure of the domain. In what follows  $\partial_p Q = (\{T\} \times \mathcal{O}) \cup ([0, T] \times \partial\mathcal{O})$  denotes the parabolic boundary of  $Q$ .

**Theorem 4.1.** (1) Assume that  $(\mathcal{A}_1)$ – $(\mathcal{A}_5)$  hold. If the triplet  $(u, v, \mu)$  is a solution to the RBSPDE (1.1), then

$$\begin{aligned} & \text{esssup}_{(\omega, t, x) \in \Omega \times Q} u^\pm \\ & \leq C \left( \text{esssup}_{(\omega, t, x) \in \Omega \times \partial_p Q} u^\pm + \text{esssup}_{(\omega, t, x) \in \Omega \times \partial_p Q} \hat{\xi}^\pm \right. \\ & \quad \left. + A_p(f_0^\pm, g_0; Q) + B_2(f_0^\pm, g_0; Q) + A_p(\hat{f}^\pm, \hat{g}; Q) + B_2(\hat{f}^\pm, \hat{g}; Q) \right), \end{aligned}$$

where the constant  $C$  depends only on  $\lambda, \kappa, \beta, L, \varrho, T, p$  and  $n$ .

(2) If all conditions in (1) hold except assumption  $(\mathcal{A}_5)$  is changed to

$$g(t, x, r, 0, 0) = g(t, x, 0, 0) \text{ and } f(t, x, r, 0, 0) \text{ is non-increasing w.r.t. } r, \quad (4.1)$$

then

$$\begin{aligned} & \text{esssup}_{(\omega, t, x) \in \Omega \times Q} u^\pm \\ & \leq \text{esssup}_{(\omega, t, x) \in \Omega \times \partial_p Q} u^\pm + \text{esssup}_{(\omega, t, x) \in \Omega \times \partial_p Q} \hat{\xi}^\pm \\ & \quad + C \left( A_p(f_0^\pm, g_0; Q)^{\frac{np}{np+2(p-n-2)}} B_2(f_0^\pm, g_0; Q)^{\frac{2(p-n-2)}{np+2(p-n-2)}} \right. \\ & \quad \left. + A_p(\hat{f}^\pm, \hat{g}; Q)^{\frac{np}{np+2(p-n-2)}} B_2(\hat{f}^\pm, \hat{g}; Q)^{\frac{2(p-n-2)}{np+2(p-n-2)}} \right), \end{aligned}$$

where  $C$  depends on  $\lambda, \kappa, \beta, L, \varrho, n, p$  and  $T$ .

*Proof.* We only consider the estimate for the positive part  $u^+$ . The one for the negative part  $u^-$  follows analogously. Further, we may w.l.o.g. assume that  $f(t, x, r, 0, 0)$  is non-increasing in  $r$ . Otherwise, the desired maximum principle can be derived from the maximum principle for the RBSPDE

$$\left\{ \begin{array}{l} -d\bar{u}(t, x) = [\partial_j(a^{ij}(t, x)\partial_i\bar{u}(t, x) + \sigma^{jr}(t, x)\bar{v}^r(t, x)) + \bar{f}(t, x, \bar{u}(t, x), \nabla\bar{u}(t, x), \bar{v}(t, x)) \\ \quad + \nabla \cdot \bar{g}(t, x, \bar{u}(t, x), \nabla\bar{u}(t, x), \bar{v}(t, x))] dt + \bar{\mu}(dt, x) - \bar{v}^r(t, x) dW_t^r, \\ \bar{u}(T, x) = \bar{G}(x), \\ \bar{u}(t, x) \geq \bar{\xi}(t, x) \quad dt \times dx \times d\mathbb{P} - a.e., \\ \int_Q (\bar{u}(t, x) - \bar{\xi}(t, x)) \bar{\mu}(dt, dx) = 0, \end{array} \right.$$

where  $\bar{u}(t, x) = e^{Lt}u(t, x)$ ,  $\bar{v}(t, x) = e^{Lt}v(t, x)$ ,  $\bar{\mu}(dt, dx) = e^{Lt}\mu(dt, dx)$ ,  $\bar{G}(x) = e^{LT}G(x)$ ,  $\bar{\xi}(t, x) = e^{Lt}\xi(t, x)$  and

$$\begin{aligned} \bar{f}(t, x, \bar{u}(t, x), \nabla\bar{u}(t, x), \bar{v}(t, x)) &= e^{Lt}f(t, x, e^{-Lt}\bar{u}(t, x), e^{-Lt}\nabla\bar{u}(t, x), e^{-Lt}\bar{v}(t, x)) - L\bar{u}(t, x) \\ \bar{g}(t, x, \bar{u}(t, x), \nabla\bar{u}(t, x), \bar{v}(t, x)) &= e^{Lt}g(t, x, e^{-Lt}\bar{u}(t, x), e^{-Lt}\nabla\bar{u}(t, x), e^{-Lt}\bar{v}(t, x)). \end{aligned}$$

Now, for  $t \in [0, T]$  define

$$\bar{k} = \text{esssup}_{(\omega, t, x) \in \Omega \times \partial_p \mathcal{O}_t} u^+ + \text{esssup}_{(\omega, t, x) \in \Omega \times \mathcal{O}_t} \hat{\xi}^+.$$

For a positive constant  $k$  to be determined later and each  $m \in \mathbb{N}_0$ , let  $\bar{k}_m = k(1 - 2^{-m})$  and  $k_m = \bar{k}_m + \bar{k}$ . By Theorem A.5, for  $m \geq 1$ ,

$$\begin{aligned} & \| (u - k_m)^+(t) \|^2 + \int_t^T \| v^{k_m}(s) \|^2 ds \\ &= -2 \int_t^T \langle \partial_j(u - k_m)^+(s), a^{ij}\partial_i(u - k_m)^+(s) + \sigma^{jr}(s)v^{k_m, r}(s) \rangle ds \\ & \quad - 2 \int_t^T \langle \partial_j(u - k_m)^+(s), g^{j, k_m}(s, (u - k_m)^+(s), \nabla u(s), v^{k_m}(s)) \rangle ds \\ & \quad + 2 \int_t^T \langle (u - k_m)^+(s), f^{k_m}(s, (u - k_m)^+(s), \nabla u(s), v^{k_m}(s)) \rangle ds \\ & \quad + 2 \int_{\mathcal{O}_t} (u - k_m)^+(s, x) \mu(ds, dx) - 2 \int_t^T \langle (u - k_m)^+(s), v^{r, k_m}(s) dW_s^r \rangle, \end{aligned} \tag{4.2}$$

where  $v^{r, k_m} := v^r 1_{\{u > k_m\}}$ ,  $f^{k_m}(\cdot, \cdot, \cdot, X, \cdot, \cdot) := f(\cdot, \cdot, \cdot, X + k_m, \cdot, \cdot)$ ,  $g^{j, k_m}(\cdot, \cdot, \cdot, X, \cdot, \cdot) := g^j(\cdot, \cdot, \cdot, X + k_m, \cdot, \cdot)$ . All terms in (4.2) are well defined. In particular, the stochastic integral is in fact a martingale. Taking conditional expectations on both sides w.r.t.  $\mathcal{F}_t$  yields the following estimates for the remaining terms. Similar estimates as for  $I_1$  in the proof of Theorem 3.2 yield,

$$\begin{aligned} J_1 &:= -2E \left[ \int_t^T \langle \partial_j(u - k_m)^+(s), a^{ij}\partial_i(u - k_m)^+(s) + \sigma^{jr}(s)v^{r, k_m}(s) \rangle ds | \mathcal{F}_t \right] \\ &\leq -\lambda E \left[ \int_t^T \| \nabla(u - k_m)^+(s) \|^2 ds | \mathcal{F}_t \right] + \frac{1}{\varrho} E \left[ \int_t^T \| v^{k_m}(s) \|^2 ds | \mathcal{F}_t \right]. \end{aligned} \tag{4.3}$$

By analogy to the estimate of  $I_2$ , for each  $\epsilon > 0$  and  $\theta > 0$ , we have that

$$\begin{aligned}
J_2 &:= -2E \left[ \int_t^T \langle \partial_j (u - k_m)^+(s), g^{j,k_m}(s, (u - k_m)^+(s), \nabla u(s), v^{k_m}(s)) \rangle ds | \mathcal{F}_t \right] \\
&\leq 2E \left[ \int_t^T \langle |\nabla(u - k_m)^+(s)|, |g_0^{k_m}| + L|(u - k_m)^+(s)| + \frac{\kappa}{2} |\nabla(u - k_m)^+(s)| \right. \\
&\quad \left. + \sqrt{\beta} |v^{k_m}(s)| \rangle ds | \mathcal{F}_t \right] \\
&\leq (\kappa + \beta\theta + \epsilon) E \left[ \int_t^T \|\nabla(u - k_m)^+(s)\|^2 ds | \mathcal{F}_t \right] + \frac{1}{\theta} E \left[ \int_t^T \|v^{k_m}(s)\|^2 ds | \mathcal{F}_t \right] \\
&\quad + \frac{L^2}{\epsilon} E \left[ \int_t^T \|(u - k_m)^+(s)\|^2 ds | \mathcal{F}_t \right] + 2E \left[ \int_t^T \langle |\nabla(u - k_m)^+(s)|, |g_0^{k_m}(s)| \rangle ds | \mathcal{F}_t \right].
\end{aligned} \tag{4.4}$$

From

$$[(u - k_{m-1})^+ - (u - k_m)^+] 1_{\{u > k_m\}} = (k_m - k_{m-1}) 1_{\{u > k_m\}} = 2^{-m} k 1_{\{u > k_m\}}$$

we get

$$1_{\{u > k_m\}} \leq \frac{2^m (u - k_{m-1})^+}{k} 1_{\{u > k_m\}} \leq \frac{2^m (u - k_{m-1})^+}{k}. \tag{4.5}$$

By (4.5) and  $(\mathcal{A}_5)$  it holds that:

$$\begin{aligned}
&E \left[ \int_t^T \langle |\nabla(u - k_m)^+(s)|, |g_0^{k_m}(s)| \rangle ds | \mathcal{F}_t \right] \\
&\leq \left( E \left[ \int_t^T \|\nabla(u - k_m)^+(s)\|^2 ds | \mathcal{F}_t \right] \right)^{\frac{1}{2}} \left( E \left[ \int_t^T \int_{\mathcal{O}} |g_0^{k_m}(s, x) 1_{\{u > k_m\}}|^2 dx ds | \mathcal{F}_t \right] \right)^{\frac{1}{2}} \\
&\leq \|\nabla(u - k_m)^+\|_{0,2;\mathcal{O}_t} \|g_0^{k_m}\|_{0,p;\mathcal{O}_t} \left( E \left[ \int_t^T \int_{\mathcal{O}} 1_{\{u > k_m\}} dx ds | \mathcal{F}_t \right] \right)^{\frac{1}{2} - \frac{1}{p}} \\
&\leq \|\nabla(u - k_m)^+\|_{0,2;\mathcal{O}_t} \|g_0^{k_m}\|_{0,p;\mathcal{O}_t} \left( E \left[ \int_t^T \int_{\mathcal{O}} \left( \frac{2^m (u - k_{m-1})^+}{k} \right)^{\frac{2(n+2)}{n}} dx ds | \mathcal{F}_t \right] \right)^{\frac{1}{2} - \frac{1}{p}} \\
&\leq \left( \frac{2^m}{k} \right)^{1 + \frac{2(p-n-2)}{np}} \|\nabla(u - k_m)^+\|_{0,2;\mathcal{O}_t} \|g_0^{k_m}\|_{0,p;\mathcal{O}_t} \|(u - k_{m-1})^+\|_{0, \frac{2(n+2)}{n}; \mathcal{O}_t}^{1 + \frac{2(p-n-2)}{np}} \\
&\leq \left( \frac{2^m}{k} \right)^{1 + \frac{2(p-n-2)}{np}} \|\nabla(u - k_{m-1})^+\|_{0,2;\mathcal{O}_t} \|g_0^{k_m}\|_{0,p;\mathcal{O}_t} \|(u - k_{m-1})^+\|_{0, \frac{2(n+2)}{n}; \mathcal{O}_t}^{1 + \frac{2(p-n-2)}{np}} \\
&\leq \left( \frac{2^m}{k} \right)^{1 + \frac{2(p-n-2)}{np}} \|\nabla(u - k_{m-1})^+\|_{0,2;\mathcal{O}_t} (\|g_0\|_{0,p;\mathcal{O}_t} + Lk_m) \|(u - k_{m-1})^+\|_{0, \frac{2(n+2)}{n}; \mathcal{O}_t}^{1 + \frac{2(p-n-2)}{np}}.
\end{aligned} \tag{4.6}$$

Combining (4.4) and (4.6), we see that

$$\begin{aligned}
J_2 &\leq (\kappa + \beta\theta + \epsilon) E \left[ \int_t^T \|\nabla(u - k_m)^+(s)\|^2 ds | \mathcal{F}_t \right] + \frac{1}{\theta} E \left[ \int_t^T \|v^{k_m}(s)\|^2 ds | \mathcal{F}_t \right] \\
&\quad + \frac{L^2}{\epsilon} E \left[ \int_t^T \|(u - k_m)^+(s)\|^2 ds | \mathcal{F}_t \right] \\
&\quad + 2 \left( \frac{2^m}{k} \right)^{1 + \frac{2(p-n-2)}{np}} \|\nabla(u - k_{m-1})^+\|_{0,2;\mathcal{O}_t} (\|g_0\|_{0,p;\mathcal{O}_t} + Lk_m) \|(u - k_{m-1})^+\|_{0, \frac{2(n+2)}{n}; \mathcal{O}_t}^{1 + \frac{2(p-n-2)}{np}}.
\end{aligned} \tag{4.7}$$

For each  $\epsilon_1 > 0$ , by (2.2) the monotonicity of  $f(t, x, r, 0, 0)$  yields:

$$\begin{aligned}
J_3 &:= 2E \left[ \int_t^T \langle (u - k_m)^+(s), f^{k_m}(s, (u - k_m)^+(s), \nabla u(s), v^{k_m}(s)) \rangle ds | \mathcal{F}_t \right] \\
&\leq 2E \left[ \int_t^T \langle (u - k_m)^+(s), f_0^{k_m}(s) + L(u - k_m)^+(s) + L\nabla(u - k_m)^+(s) + L|v^{k_m}(s)| \rangle ds | \mathcal{F}_t \right] \\
&\leq 2E \left[ \int_t^T \langle (u - k_m)^+(s), f_0(s) + L(u - k_m)^+(s) + L\nabla(u - k_m)^+(s) + L|v^{k_m}(s)| \rangle ds | \mathcal{F}_t \right] \quad (4.8) \\
&\leq \left( 2L + \frac{2L^2}{\epsilon_1} \right) E \left[ \int_t^T \|(u - k_m)^+(s)\|^2 ds | \mathcal{F}_t \right] + \epsilon_1 E \left[ \int_t^T \|\nabla(u - k_m)^+(s)\|^2 ds | \mathcal{F}_t \right] \\
&\quad + \epsilon_1 E \left[ \int_t^T \|v^{k_m}(s)\|^2 ds | \mathcal{F}_t \right] + 2E \left[ \int_t^T \langle (u - k_m)^+(s), f_0(s) \rangle ds | \mathcal{F}_t \right].
\end{aligned}$$

By (4.5) again, we have

$$\begin{aligned}
&E \left[ \int_t^T \langle (u - k_m)^+(s), f_0(s) \rangle ds | \mathcal{F}_t \right] \\
&\leq E \left[ \int_t^T \langle (u - k_m)^+(s), f_0^+(s) \rangle ds | \mathcal{F}_t \right] \\
&\leq \left( E \left[ \int_t^T \int_{\mathcal{O}} |(u - k_m)^+|^{\frac{2(n+2)}{n}} dx ds | \mathcal{F}_t \right] \right)^{\frac{n}{2(n+2)}} \left( E \left[ \int_t^T \int_{\mathcal{O}} |f_0^+(s, x)|^{\frac{p(n+2)}{p+n+2}} dx ds | \mathcal{F}_t \right] \right)^{\frac{p+n+2}{p(n+2)}} \\
&\quad \times \left( E \left[ \int_t^T \int_{\mathcal{O}} 1_{\{u > k_m\}} dx ds | \mathcal{F}_t \right] \right)^{\frac{1}{2} - \frac{1}{p}} \quad (4.9) \\
&\leq \|(u - k_m)^+\|_{0, \frac{2(n+2)}{n}; \mathcal{O}_t} \|f_0^+\|_{0, \frac{p(n+2)}{p+n+2}; \mathcal{O}_t} \left( E \left[ \int_t^T \int_{\mathcal{O}} \left( \frac{2^m(u - k_{m-1})^+}{k} \right)^{\frac{2(n+2)}{n}} dx ds | \mathcal{F}_t \right] \right)^{\frac{1}{2} - \frac{1}{p}} \\
&\leq \left( \frac{2^m}{k} \right)^{1 + \frac{2(p-n-2)}{np}} \|(u - k_m)^+\|_{0, \frac{2(n+2)}{n}; \mathcal{O}_t} \|(u - k_{m-1})^+\|_{0, \frac{2(n+2)}{n}; \mathcal{O}_t}^{1 + \frac{2(p-n-2)}{np}} \|f_0^+\|_{0, \frac{p(n+2)}{p+n+2}; \mathcal{O}_t} \\
&\leq \left( \frac{2^m}{k} \right)^{1 + \frac{2(p-n-2)}{np}} \|(u - k_{m-1})^+\|_{0, \frac{2(n+2)}{n}; \mathcal{O}_t}^{2 + \frac{2(p-n-2)}{np}} \|f_0^+\|_{0, \frac{p(n+2)}{p+n+2}; \mathcal{O}_t}.
\end{aligned}$$

Therefore, by (4.8) and (4.9) we conclude

$$\begin{aligned}
J_3 &\leq \left( 2L + \frac{2L^2}{\epsilon_1} \right) E \left[ \int_t^T \|(u - k_m)^+(s)\|^2 ds | \mathcal{F}_t \right] + \epsilon_1 E \left[ \int_t^T \|\nabla(u - k_m)^+(s)\|^2 ds | \mathcal{F}_t \right] \\
&\quad + \epsilon_1 E \left[ \int_t^T \|v^{k_m}(s)\|^2 ds | \mathcal{F}_t \right] + 2 \left( \frac{2^m}{k} \right)^{1 + \frac{2(p-n-2)}{np}} \|(u - k_{m-1})^+\|_{0, \frac{2(n+2)}{n}; \mathcal{O}_t}^{2 + \frac{2(p-n-2)}{np}} \|f_0^+\|_{0, \frac{p(n+2)}{p+n+2}; \mathcal{O}_t}. \quad (4.10)
\end{aligned}$$

Finally, note that

$$\int_t^T \int_{\mathcal{O}} (u - k_m)^+ \mu(dx ds) \leq \int_t^T \int_{\mathcal{O}} (u - \xi)^+ \mu(dx ds) + \int_t^T \int_{\mathcal{O}} (\xi - \hat{\xi}^+)^+ \mu(dx ds) = 0.$$

Combining the above estimates, we get

$$\|(u - k_m)^+(t)\|^2 + E \left[ \int_t^T \|v^{k_m}(s)\|^2 ds | \mathcal{F}_t \right]$$

$$\begin{aligned}
&\leq (-\lambda + \kappa + \beta\theta + \epsilon + \epsilon_1) E \left[ \int_t^T \|\nabla(u - k_m)^+\|^2 ds | \mathcal{F}_t \right] \\
&\quad + \left( \frac{1}{\theta} + \frac{1}{\varrho} + \epsilon_1 \right) E \left[ \int_t^T \|v^{k_m}\|^2 ds | \mathcal{F}_t \right] + \left( 2L + \frac{L^2}{\epsilon} + \frac{2L^2}{\epsilon_1} \right) E \left[ \int_t^T \|(u - k_m)^+\|^2 ds | \mathcal{F}_t \right] \\
&\quad + 2 \left( \frac{2^m}{k} \right)^{1 + \frac{2(p-n-2)}{np}} \|\nabla(u - k_{m-1})^+\|_{0,2;\mathcal{O}_t} \|(u - k_{m-1})^+\|_{0, \frac{2(n+2)}{n}; \mathcal{O}_t}^{1 + \frac{2(p-n-2)}{np}} (\|g_0\|_{0,p;\mathcal{O}_t} + Lk_m) \\
&\quad + 2 \left( \frac{2^m}{k} \right)^{1 + \frac{2(p-n-2)}{np}} \|(u - k_{m-1})^+\|_{0, \frac{2(n+2)}{n}; \mathcal{O}_t}^{2 + \frac{2(p-n-2)}{np}} \|f_0^+\|_{0, \frac{p(n+2)}{p+n+2}; \mathcal{O}_t}.
\end{aligned}$$

From this it is straightforward to see that

$$\begin{aligned}
&\min\{1, \lambda - \kappa - \beta\theta - \epsilon - \epsilon_1\} \left\{ \|(u - k_m)^+(t)\|^2 + E \left[ \int_t^T \|\nabla(u - k_m)^+\|^2 ds | \mathcal{F}_t \right] \right\} \\
&\quad + \left( 1 - \frac{1}{\theta} - \frac{1}{\varrho} - \epsilon_1 \right) E \left[ \int_t^T \|v^{k_m}\|^2 ds | \mathcal{F}_t \right] \\
&\leq \left( 2L + \frac{L^2}{\epsilon} + \frac{2L^2}{\epsilon_1} \right) E \left[ \int_t^T \|(u - k_m)^+\|^2 ds | \mathcal{F}_t \right] \\
&\quad + 2 \left( \frac{2^m}{k} \right)^{1 + \frac{2(p-n-2)}{np}} \|\nabla(u - k_{m-1})^+\|_{0,2;\mathcal{O}_t} \|(u - k_{m-1})^+\|_{0, \frac{2(n+2)}{n}; \mathcal{O}_t}^{1 + \frac{2(p-n-2)}{np}} (\|g_0\|_{0,p;\mathcal{O}_t} + Lk_m) \\
&\quad + 2 \left( \frac{2^m}{k} \right)^{1 + \frac{2(p-n-2)}{np}} \|(u - k_{m-1})^+\|_{0, \frac{2(n+2)}{n}; \mathcal{O}_t}^{2 + \frac{2(p-n-2)}{np}} \|f_0^+\|_{0, \frac{p(n+2)}{p+n+2}; \mathcal{O}_t}.
\end{aligned}$$

By assumption  $(\mathcal{A}_2)$ , there exists  $\theta > \varrho'$  such that  $\lambda - \kappa - \theta\beta > 0$  and  $\frac{1}{\theta} + \frac{1}{\varrho} < 1$ . So, we can take  $\epsilon$  and  $\epsilon_1$  small enough such that  $\lambda - \kappa - \beta\theta - \epsilon - \epsilon_1 > 0$  and  $1 - \frac{1}{\theta} - \frac{1}{\varrho} - \epsilon_1 > 0$ . Taking the supremum on both sides, Lemma A.3 yields,

$$\begin{aligned}
&\|(u - k_m)^+\|_{\mathcal{V}_2(\mathcal{O}_t)}^2 + \|v^{k_m}\|_{0,2;\mathcal{O}_t}^2 \\
&\leq C_1(\lambda, \kappa, \beta, L, \theta, \varrho, \epsilon, \epsilon_1) \int_t^T \|(u - k_m)^+\|_{\mathcal{V}_2(\mathcal{O}_s)}^2 ds \\
&\quad + C_1(\lambda, \kappa, \beta, L, \theta, \varrho, n, \epsilon, \epsilon_1) \left( \frac{2^m}{k} \right)^{1 + \frac{2(p-n-2)}{np}} \|(u - k_{m-1})^+\|_{\mathcal{V}_2(\mathcal{O}_t)}^{2 + \frac{2(p-n-2)}{np}} (A_p(f_0^+, g_0; \mathcal{O}_t) + Lk_m).
\end{aligned}$$

Gronwall's inequality yields that

$$\begin{aligned}
&\|(u - k_m)^+\|_{\mathcal{V}_2(\mathcal{O}_t)}^2 + \|v^{k_m}\|_{0,2;\mathcal{O}_t}^2 \\
&\leq C_2(\lambda, \kappa, \beta, L, \theta, \varrho, T, n, \epsilon, \epsilon_1) \left( \frac{2^m}{k} \right)^{1 + \frac{2(p-n-2)}{np}} \|(u - k_{m-1})^+\|_{\mathcal{V}_2(\mathcal{O}_t)}^{2 + \frac{2(p-n-2)}{np}} (A_p(f_0^+, g_0; \mathcal{O}_t) + Lk_m).
\end{aligned} \tag{4.11}$$

Letting  $k \geq \bar{k} + \frac{A_p(f_0^+, g_0; \mathcal{O}_t)}{L}$ , it follows from (4.11) that

$$\begin{aligned}
&\|(u - k_m)^+\|_{\mathcal{V}_2(\mathcal{O}_t)}^2 + \|v^{k_m}\|_{0,2;\mathcal{O}_t}^2 \\
&\leq C_3(\lambda, \kappa, \beta, L, \theta, \varrho, T, n, \epsilon, \epsilon_1) \frac{2^{1 + \frac{2(p-n-2)}{np}}}{k^{\frac{2(p-n-2)}{np}}} \left( 2^{1 + \frac{2(p-n-2)}{np}} \right)^{m-1} \|(u - k_{m-1})^+\|_{\mathcal{V}_2(\mathcal{O}_t)}^{2 + \frac{2(p-n-2)}{np}}.
\end{aligned} \tag{4.12}$$

In terms of  $a_m := \|(u - k_m)^+\|_{\mathcal{V}_2(\mathcal{O}_t)}^2$ ,  $C_0 := C_3(\lambda, \kappa, \beta, L, \theta, \varrho, T, \epsilon, \epsilon_1) \frac{2^{1+\frac{2(p-n-2)}{np}}}{k^{\frac{2(p-n-2)}{np}}} > 0$ ,  $b := 2^{1+\frac{2(p-n-2)}{np}} > 1$  and  $\delta := \frac{(p-n-2)}{np} > 0$ , we get that

$$a_m \leq C_0 b^{m-1} a_{m-1}^{1+\delta}.$$

Now, let

$$k \geq C_3(\lambda, \kappa, \beta, L, \theta, \varrho, T, \epsilon, \epsilon_1)^{\frac{1}{2\delta}} 2^{(1+2\delta)(\frac{1}{2\delta^2} + \frac{1}{2\delta})} \|(u - \bar{k})^+\|_{\mathcal{V}_2(\mathcal{O}_t)}.$$

Then  $a_0 \leq C_0^{-\frac{1}{\delta}} b^{-\frac{1}{\delta^2}}$ . Therefore, Lemma A.1 can be applied to get  $\lim_{m \rightarrow \infty} a_m = 0$ . Along with the above estimates for  $k$  this implies that

$$\text{esssup}_{(\omega, t, x) \in \Omega \times \mathcal{O}_t} (u - \bar{k})^+ \leq C (\bar{k} + A_p(f_0^+, g_0; \mathcal{O}_t) + \|(u - \bar{k})^+\|_{\mathcal{V}_2(\mathcal{O}_t)}), \quad (4.13)$$

where  $C$  depends on  $\lambda, \kappa, \beta, L, \varrho, T, p$  and  $n$ . The estimates of terms  $\|(u - \bar{k})^+\|_{\mathcal{V}_2(\mathcal{O}_t)}$  and  $\text{esssup}_{(\omega, t, x) \in \Omega \times \mathcal{O}_t} \hat{\xi}^+$  are given in the following Lemma 4.3(1) and Lemma 4.4(1). Finally we arrive at

$$\begin{aligned} & \text{esssup}_{(\omega, t, x) \in \Omega \times \mathcal{O}_t} u^+ \\ & \leq C \left( \text{esssup}_{(\omega, t, x) \in \Omega \times \partial_p \mathcal{O}_t} u^+ + \text{esssup}_{(\omega, t, x) \in \Omega \times \partial_p \mathcal{O}_t} \hat{\xi}^+ \right. \\ & \quad \left. + A_p(f_0^+, g_0; \mathcal{O}_t) + B_2(f_0^+, g_0; \mathcal{O}_t) + A_p(\hat{f}^+, \hat{g}; \mathcal{O}_t) + B_2(\hat{f}^+, \hat{g}; \mathcal{O}_t) \right), \end{aligned} \quad (4.14)$$

where  $C$  depends on  $\lambda, \kappa, \beta, L, \varrho, T, p$  and  $n$ .

(2) For each  $t \in [0, T]$ , let  $k \geq \text{esssup}_{(\omega, t, x) \in \Omega \times \partial_p \mathcal{O}_t} u^+ + \text{esssup}_{(\omega, t, x) \in \Omega \times \mathcal{O}_t} \hat{\xi}^+$ . By Theorem A.5 we obtain

$$\begin{aligned} & \|(u - k)^+(t)\|^2 + \int_t^T \|v^k(s)\|^2 ds \\ & = -2 \int_t^T \langle \partial_j (u - k)^+(s), a^{ij} \partial_i (u - k)^+(s) + \sigma^{jr}(s) v^{k,r}(s) \rangle ds \\ & \quad - 2 \int_t^T \langle \partial_j (u - k)^+(s), g^{j,k}(s, (u - k)^+(s), \nabla u(s), v^k(s)) \rangle ds \\ & \quad + 2 \int_t^T \langle (u - k)^+(s), f^k(s, (u - k)^+(s), \nabla u(s), v^k(s)) \rangle ds + 2 \int_{\mathcal{O}_t} (u - k)^+(s, x) \mu(ds, dx) \\ & \quad - 2 \int_t^T \langle (u - k)^+(s), v^{r,k}(s) dW_s^r \rangle. \end{aligned}$$

For every  $k > l \geq \text{esssup}_{(\omega, t, x) \in \Omega \times \partial_p \mathcal{O}_t} u^+ + \text{esssup}_{(\omega, t, x) \in \Omega \times \mathcal{O}_t} \hat{\xi}^+$ , we get

$$((u - l)^+ - (u - k)^+) 1_{(u > k)} = (k - l) 1_{(u > k)},$$

which implies

$$1_{(u > k)} \leq \frac{(u - l)^+}{k - l}.$$

By the assumptions on  $g$  and  $f$ , and using the same arguments in (4.3), (4.4) and (4.6)-(4.11) we obtain that

$$\|(u - k)^+\|_{\mathcal{V}_2(\mathcal{O}_t)}^2 + \|v^k\|_{0,2;\mathcal{O}_t}^2 \leq C \frac{A_p(f_0^+, g_0; \mathcal{O}_t)}{(k - l)^{1+\frac{2(p-n-2)}{np}}} \|(u - l)^+\|_{\mathcal{V}_2(\mathcal{O}_t)}^{2+\frac{2(p-n-2)}{np}},$$

where  $C$  depends on  $\lambda, \kappa, \beta, L, \varrho, T$  and  $n$ . By setting  $\phi(k) := \|(u - k)^+\|_{\mathcal{V}_2(\mathcal{O}_t)}^2$ ,  $\alpha := 1 + \frac{2(p-n-2)}{np} > 0$ ,  $\zeta := 1 + \frac{p-n-2}{np}$  and  $C_1 := C A_p(f_0^+, g_0; \mathcal{O}_t)$ , the following statement holds for each  $k > l \geq \text{esssup}_{(\omega, t, x) \in \Omega \times \partial_p \mathcal{O}_t} u^+ + \text{esssup}_{(\omega, t, x) \in \Omega \times \mathcal{O}_t} \hat{\xi}^+$ :

$$\phi(k) \leq \frac{C_1}{(k - l)^\alpha} \phi(l)^\zeta.$$

If we define  $d := C_1^{\frac{1}{\alpha}} \left| \phi(\text{esssup}_{(\omega,t,x) \in \Omega \times \partial_p \mathcal{O}_t} u^+ + \text{esssup}_{(\omega,t,x) \in \Omega \times \mathcal{O}_t} \hat{\xi}^+) \right|^{\frac{\zeta-1}{\alpha}} 2^{\frac{1+\alpha}{\alpha}}$ , then by Corollary A.2,

$$\|(u - d - \text{esssup}_{(\omega,t,x) \in \Omega \times \partial_p \mathcal{O}_t} u^+ - \text{esssup}_{(\omega,t,x) \in \Omega \times \mathcal{O}_t} \hat{\xi}^+)^+\|_{V_2(\mathcal{O}_t)} = 0,$$

and so Lemma 4.3(2) yields

$$\begin{aligned} \text{esssup}_{(\omega,t,x) \in \Omega \times \mathcal{O}_t} u^+ &\leq \text{esssup}_{(\omega,t,x) \in \Omega \times \partial_p \mathcal{O}_t} u^+ + \text{esssup}_{(\omega,t,x) \in \Omega \times \mathcal{O}_t} \hat{\xi}^+ \\ &\quad + C A_p(f_0^+, g_0; \mathcal{O}_t)^{\frac{1}{\alpha}} B_2(f_0^+, g_0; \mathcal{O}_t)^{\frac{2(\zeta-1)}{\alpha}}, \end{aligned} \quad (4.15)$$

where  $C$  depends on  $\lambda, \kappa, \beta, L, \varrho, n, p$  and  $T$ . Therefore (4.15) and (4.19) yield

$$\begin{aligned} \text{esssup}_{(\omega,t,x) \in \Omega \times \mathcal{O}_t} u^+ &\leq \text{esssup}_{(\omega,t,x) \in \Omega \times \partial_p \mathcal{O}_t} u^+ + \text{esssup}_{(\omega,t,x) \in \Omega \times \partial_p \mathcal{O}_t} \hat{\xi}^+ \\ &\quad + C A_p(\hat{f}^+, \hat{g}; \mathcal{O}_t)^{\frac{1}{\alpha}} B_2(\hat{f}^+, \hat{g}; \mathcal{O}_t)^{\frac{2(\zeta-1)}{\alpha}} \\ &\quad + C A_p(f_0^+, g_0; \mathcal{O}_t)^{\frac{1}{\alpha}} B_2(f_0^+, g_0; \mathcal{O}_t)^{\frac{2(\zeta-1)}{\alpha}}. \end{aligned}$$

□

When the domain  $\mathcal{O}$  is bounded,  $\|\cdot\|_{0,2;Q}$  can be bounded by  $\|\cdot\|_{0,p;Q}$  and  $\|\cdot\|_{0,\frac{p(n+2)}{p+n+2};Q}$  and we have the following maximum principle for the RBSPDE (1.1) on a bounded domain.

**Corollary 4.2.** (1) Assume  $(\mathcal{A}_1)$ – $(\mathcal{A}_5)$  hold and  $\mathcal{O}$  is bounded. If the triplet  $(u, v, \mu)$  is a solution to the RBSPDE (1.1), then

$$\begin{aligned} &\text{esssup}_{(\omega,t,x) \in \Omega \times Q} u^\pm \\ &\leq C \left( \text{esssup}_{(\omega,t,x) \in \Omega \times \partial_p Q} u^\pm + \text{esssup}_{(\omega,t,x) \in \Omega \times \partial_p Q} \hat{\xi}^\pm \right. \\ &\quad \left. + A_p(f_0^\pm, g_0; Q) + A_p(\hat{f}^\pm, \hat{g}; Q) \right), \end{aligned}$$

where the constant  $C$  depends only on  $\lambda, \kappa, \beta, L, \varrho, T, p, n$  and  $|\mathcal{O}|$ .

(2) Suppose that  $(\mathcal{A}_1)$ – $(\mathcal{A}_4)$  and (4.1) hold. Then for each solution  $(u, v, \mu)$  to the RBSPDE (1.1), it holds true that

$$\begin{aligned} &\text{esssup}_{(\omega,t,x) \in \Omega \times Q} u^\pm \\ &\leq \text{esssup}_{(\omega,t,x) \in \Omega \times \partial_p Q} u^\pm + \text{esssup}_{(\omega,t,x) \in \Omega \times \partial_p Q} \hat{\xi}^\pm \\ &\quad + C \left( A_p(f_0^\pm, g_0; Q) + A_p(\hat{f}^\pm, \hat{g}; Q) \right), \end{aligned}$$

where  $C$  depends on  $\lambda, \kappa, \beta, L, \varrho, T, p, n$ , and  $|\mathcal{O}|$ .

**Lemma 4.3.** (1) Under the same conditions as in Theorem 4.1(1), for each  $t \in [0, T]$  and each  $k \geq \text{esssup}_{(\omega,t,x) \in \Omega \times \partial_p \mathcal{O}_t} u^+ + \text{esssup}_{(\omega,t,x) \in \Omega \times \mathcal{O}_t} \hat{\xi}^+$ , we have

$$\|(u - k)^+\|_{V_2(\mathcal{O}_t)} \leq C(B_2(f_0^+, g_0; \mathcal{O}_t) + k),$$

where  $C$  depends on  $\lambda, \kappa, \beta, L, \varrho$  and  $T$ .

(2) Under the same conditions as in Theorem 4.1(2), for each  $t \in [0, T]$  and  $k \geq \text{esssup}_{(\omega,t,x) \in \Omega \times \partial_p \mathcal{O}_t} u^+ + \text{esssup}_{(\omega,t,x) \in \Omega \times \mathcal{O}_t} \hat{\xi}^+$  and we have

$$\|(u - k)^+\|_{V_2(\mathcal{O}_t)} \leq C B_2(f_0^+, g_0; \mathcal{O}_t),$$

where  $C$  depends on  $\lambda, \kappa, \beta, L, \varrho$  and  $T$ .

*Proof.* (1) As in the proof of Theorem 4.1, we may assume w.l.o.g that  $f(t, x, r, 0, 0)$  is non-increasing in  $r$ . For

$$k \geq \text{esssup}_{(\omega, t, x) \in \Omega \times \partial_p \mathcal{O}_t} u^+ + \text{esssup}_{(\omega, t, x) \in \Omega \times \mathcal{O}_t} \hat{\xi}^+,$$

we have

$$\int_t^T \int_{\mathcal{O}} (u - k)^+ \mu(dx ds) \leq \int_t^T \int_{\mathcal{O}} (u - \xi)^+ \mu(dx ds) + \int_t^T \int_{\mathcal{O}} (\xi - \hat{\xi}^+)^+ \mu(dx ds) = 0.$$

Applying Theorem A.5, we have

$$\begin{aligned} & \| (u - k)^+(t) \|^2 + E \left( \int_t^T \| v^k(s) \|^2 ds | \mathcal{F}_t \right) \\ & \leq -2E \left( \int_t^T \langle \partial_j (u - k)^+(s), a^{ij} \partial_i (u - k)^+(s) + \sigma^{jr}(s) v^{k,r}(s) \rangle ds | \mathcal{F}_t \right) \\ & \quad - 2E \left( \int_t^T \langle \partial_j (u - k)^+(s), g^{j,k}(s, (u - k)^+(s), \nabla u(s), v^k(s)) \rangle ds | \mathcal{F}_t \right) \\ & \quad + 2E \left( \int_t^T \langle (u - k)^+(s), f^k(s, (u - k)^+(s), \nabla u(s), v^k(s)) \rangle ds | \mathcal{F}_t \right) \\ & := K_1 + K_2 + K_3, \end{aligned} \tag{4.16}$$

where  $v^{r,k} := v^r 1_{\{u > k\}}$ ,  $g^{j,k}(\cdot, \cdot, \cdot, X, \cdot, \cdot) := g^j(\cdot, \cdot, \cdot, X + k, \cdot, \cdot)$ ,  $f^k(\cdot, \cdot, \cdot, X, \cdot, \cdot) := f(\cdot, \cdot, \cdot, X + k, \cdot, \cdot)$ . The quantities  $K_i$  ( $i = 1, 2, 3$ ) can now be estimated by analogy to the constants  $I_i$  ( $i = 1, 2, 3$ ) in the proof of Theorem 3.2. Specifically,  $K_1$  can be estimated as  $I_1$ , with  $u - \hat{\xi}$  and  $v - \hat{v}$  being replaced by  $(u - k)^+$  and  $v$ , respectively;  $K_2$  can be estimated as  $I_2$ , without  $\hat{g}$  (because we now have no obstacle process involved in), and  $u - \hat{\xi}$  and  $g^j(s, x, u, \nabla u, v)$  being replaced by  $(u - k)^+$  and  $g^{j,k}(s, x, (u - k)^+(s), \nabla u(s), v^k(s))$ , respectively and the estimate for  $K_3$  is similar to that for  $I_3$ , without  $\hat{f}$  and  $u - \hat{\xi}$  and  $f(s, x, u, \nabla u, v)$  being replaced by  $(u - k)^+$  and  $f^k(s, x, (u - k)^+(s), \nabla u, v^k)$ , respectively. Finally, by  $(\mathcal{A}_5)$ ,  $\|g_0^k\|_{0,2;\mathcal{O}_t}$  can be estimated by  $\|g_0\|_{0,2;\mathcal{O}_t} + Lk$ . This yields the desired result.

(2) The proof is the same as that of (1) if we note that  $\|g_0^k\|_{0,2;\mathcal{O}_t} = \|g_0\|_{0,2;\mathcal{O}_t}$  by assumption.  $\square$

The following lemma establishes the maximum principle for quasi-linear BSPDE on general domains.

**Lemma 4.4.** *Let  $(u, v)$  be a weak solution to the following quasi-linear BSPDE*

$$\begin{cases} -du(t, x) = [\partial_j (a^{ij} \partial_i u(t, x) + \sigma^{jr} v^r(t, x)) + f(t, x, u(t, x), \nabla u(t, x), v(t, x)) \\ \quad + \nabla \cdot g(t, x, u(t, x), \nabla u(t, x), v(t, x))] dt - v^r(t, x) dW_t^r, & (t, x) \in Q, \\ u(T, x) = G(x), & x \in \mathcal{O}. \end{cases} \tag{4.17}$$

(1) *If the coefficients satisfy assumptions  $(\mathcal{A}_1)$ ,  $(\mathcal{A}_2)$ ,  $(\mathcal{A}_3)$  and  $(\mathcal{A}_5)$ , then for each  $t \in [0, T]$  we have*

$$\begin{aligned} & \text{esssup}_{(\omega, t, x) \in \Omega \times \mathcal{O}_t} u^\pm \\ & \leq C \left( \text{esssup}_{(\omega, t, x) \in \Omega \times \partial_p \mathcal{O}_t} u^\pm + A_p(f_0^\pm, g_0; \mathcal{O}_t) + B_2(f_0^\pm, g_0; \mathcal{O}_t) \right) \end{aligned} \tag{4.18}$$

where  $C$  depends on  $\lambda, \kappa, \beta, L, \varrho, T, p$  and  $n$ ;

(2) *If  $(\mathcal{A}_1)$ ,  $(\mathcal{A}_2)$ ,  $(\mathcal{A}_3)$  and (4.1) hold true, then for each  $t \in [0, T]$  we have*

$$\text{esssup}_{(\omega, t, x) \in \Omega \times \mathcal{O}_t} u^\pm$$



$$\begin{aligned} &\leq \text{esssup}_{(\omega, t, x) \in \Omega \times \partial_p \mathcal{O}_t} u^\pm \\ &\quad + CA_p(f_0^\pm, g_0; \mathcal{O}_t)^{\frac{np}{np+2(p-n-2)}} B_2(f_0^\pm, g_0; \mathcal{O}_t)^{\frac{2(p-n-2)}{np+2(p-n-2)}}, \end{aligned} \quad (4.19)$$

where  $C$  depends on  $\lambda, \kappa, \beta, L, \varrho, n, p$  and  $T$ .

*Proof.* In terms of  $\bar{k} = \text{esssup}_{(\omega, t, x) \in \Omega \times \partial_p \mathcal{O}_t} u^+$  the assertion follows by establishing estimates analogous to (4.3)-(4.13) and Lemma 4.3(1).  $\square$

The proceeding lemmas allow us to establish the comparison principle for the quasi-linear BSPDE on a general domain.

**Corollary 4.5.** Let  $(u_i, v_i)$  be solutions to the quasi-linear BSPDE (4.17) with parameters  $(f_i, g, G_i, a, \sigma)$  respectively,  $i = 1, 2$ . Suppose that assumptions in Lemma 4.4 hold and that  $(u_1 - u_2)^+|_{\partial \mathcal{O}} = 0$ . Then if  $f_1(t, x, u_2, \nabla u_2, v_2) \leq f_2(t, x, u_2, \nabla u_2, v_2) \, dt \times dx \times d\mathbb{P}$ -a.e. and  $G_1 \leq G_2 \, dx \times d\mathbb{P}$ -a.e., we have  $u_1 \leq u_2 \, dt \times dx \times d\mathbb{P}$ -a.e..

*Proof.* Let  $(\underline{u}, \underline{v}) = (u_1 - u_2, v_1 - v_2)$ . Then  $(\underline{u}, \underline{v})$  is a solution to the quasi-linear BSPDE (4.17) with parameters  $(\underline{f}, \underline{g}, \underline{G}, a, \sigma)$ , where

$$\begin{aligned} \underline{f}(t, x, \cdot, \cdot, \cdot) &= f_1(t, x, \cdot + u_2, \cdot + \nabla u_2, \cdot + v_2) - f_2(t, x, u_2, \nabla u_2, v_2) \\ \underline{g}(t, x, \cdot, \cdot, \cdot) &= g(t, x, \cdot + u_2, \cdot + \nabla u_2, \cdot + v_2) - g(t, x, u_2, \nabla u_2, v_2) \\ \underline{G} &= G_1 - G_2. \end{aligned}$$

Then we have  $\underline{f}_0 := \underline{f}(\cdot, \cdot, 0, 0, 0) \leq 0$ ,  $\underline{g}_0 := \underline{g}(\cdot, \cdot, 0, 0, 0) = 0$  and  $\text{esssup}_{\Omega \times \partial_p Q} u^+ = 0$ . Therefore by Lemma 4.3 or Lemma 4.4, there holds that  $u_1 \leq u_2 \, dt \times dx \times d\mathbb{P}$ -a.e..  $\square$

## 4.2 Local Behavior of the Random Field $u^\pm$

The global maximum principle in Theorem 4.1 tells us that if the random field  $u^\pm$  is bounded on the parabolic boundary, it must be bounded in the whole domain. This section studies the local behavior of  $u^\pm$  when it is not necessarily bounded on the parabolic boundary.

**Definition 4.6.** A function  $\zeta$  is called a cut-off function on the sub-domain  $Q' \subset Q$  if it satisfies the following properties:

- (1) there exists some smooth function sequence  $\{\zeta_m\} \subset C_0^\infty(Q')$  such that  $\zeta_m, \partial_s \zeta_m$  and  $\nabla \zeta_m$  converge to  $\zeta, \partial_s \zeta$  and  $\nabla \zeta$  in  $L^\infty(Q')$  respectively;
- (2)  $\zeta \in [0, 1]$ ;
- (3) there exists a domain  $Q'' \subset \subset Q'$  and a nonempty domain  $Q''' \subset \subset Q''$  such that

$$\zeta(t, x) = \begin{cases} 0 & \text{if } (t, x) \in Q' \setminus Q'' \\ 1 & \text{if } (t, x) \in Q''', \end{cases}$$

where by  $A \subset \subset B$  we mean the closure  $\bar{A} \subseteq B$ .

We modify the definition of backward stochastic parabolic De Giorgi class in [15] as follows.

**Definition 4.7.** We say a function  $u \in \mathcal{V}_{2,0}(Q)$  belongs to a backward stochastic parabolic De Giorgi class  $BSPDG^\pm(a_0, b_0, k_0, \eta; \delta, Q)$  with

$$(a_0, b_0, k_0, \eta, \delta) \in [0, \infty) \times [0, \infty) \times [0, \infty) \times (n+2, \infty) \times (0, 1),$$

if for any  $Q_{\rho, \tau} := [t_0 - \tau, t_0] \times B_\rho(x_0) \subset Q$  with  $(\rho, \tau) \in (0, \delta] \times (0, \delta^2]$ , each cut-off function  $\zeta$  on  $Q_{\rho, \tau}$  and for each  $k \geq k_0$ , we have

$$\begin{aligned} \|\zeta(u-k)^\pm\|_{\mathcal{V}_2(Q_{\rho, \tau})}^2 &\leq b_0 \left\{ \|(u-k)^\pm\|_{0,2;Q_{\rho, \tau}}^2 \left( 1 + \|\partial_t \zeta\|_{L^\infty(Q_{\rho, \tau})} + \|\nabla \zeta\|_{L^\infty(Q_{\rho, \tau})}^2 \right) \right. \\ &\quad \left. + (k^2 + a_0^2) |(u-k)^\pm > 0|_{\infty; Q_{\rho, \tau}}^{1-\frac{2}{\eta}} \right\}, \end{aligned} \quad (4.20)$$

where  $|(u-k)^\pm > 0|_{\infty; Q_{\rho, \tau}} := \text{esssup}_{\omega \in \Omega} \sup_{s \in [t_0 - \tau, t_0]} E \left[ \int_{[s, t_0] \times B_\rho(x_0)} 1_{\{(u(t, x) - k)^\pm > 0\}} dx dt | \mathcal{F}_s \right]$ .

Here, we take  $(k, \rho, \tau) \in [k_0, \infty) \times (0, \delta] \times (0, \delta^2]$  for given  $(k_0, \delta) \in [0, \infty) \times (0, 1)$  in the above definition, instead of  $(k, \rho, \tau) \in \mathbb{R} \times (0, 1) \times (0, 1)$  as in [15, Definition 5.2]. However, a direct extension of [15, Theorem 5.8] yields the following lemma.

**Lemma 4.8.** Given  $k_0^\pm \geq 0$ , if  $u \in BSPDG^\pm(a_0^\pm, b_0^\pm, k_0^\pm, \eta; \delta, Q)$ , then

$$\text{esssup}_{(\omega, t, x) \in \Omega \times Q_{\frac{\delta}{2}}} u^\pm \leq 2k_0^\pm + C_\pm \left\{ \rho^{-\frac{n+2}{2}} \|u^\pm\|_{0,2;Q_\rho} + a_0^\pm \rho^{1-\frac{2+n}{\eta}} \right\},$$

where  $Q_\rho := [t_0 - \rho^2, t_0] \times B_\rho(x_0) \subset Q$  with  $\rho \in (0, \delta]$  and the constants  $C_\pm$  depend on  $a_0^\pm, b_0^\pm$  and  $n$ .

For the solution to the RBSPDE (1.1), we further have the following result.

**Lemma 4.9.** Let assumptions  $(\mathcal{A}_1)$ – $(\mathcal{A}_4)$  hold. Suppose  $(u, v, \mu)$  is a solution to the RBSPDE (1.1). Given  $Q_\delta := [t_0 - \delta^2, t_0] \times B_\delta(x_0) \subset Q$  with  $\delta \in (0, 1)$ , let  $k_0^\pm = \text{esssup}_{\Omega \times Q_\delta} \hat{\xi}^\pm$ . Then we have  $u \in BSPDG^\pm(a_0^\pm, b_0, k_0^\pm, \eta; \delta, Q_\delta)$  with  $\eta = p$ ,  $a_0^\pm = A_p(f_0^\pm, g_0; Q_\delta)$  and  $b_0$  depending on  $\lambda, \kappa, \beta, \varrho, \Lambda, L, n$  and  $p$ .

*Proof.* First we generalize the Itô formula to a local case for the RBSPDE (1.1). For each cut-off function  $\zeta$  on  $Q_{\rho, \tau}$  with  $(\rho, \tau) \in (0, \delta] \times (0, \delta^2]$ , we can choose a sequence of smooth functions  $\{\zeta_m\} \subset C_0^\infty(Q_{\rho, \tau})$  such that  $\zeta_m$  and its gradients w.r.t.  $s$  and  $x$  converge uniformly to  $\zeta$  and its gradient, respectively, as  $m \rightarrow \infty$ .

For  $k \geq k_0^+$ , Theorem A.5 yields that

$$\begin{aligned} &\|(u-k)^+(t)\zeta_m(t)\|_{L^2(B_\rho(x_0))}^2 + \int_t^{t_0} \|\zeta_m(s)v^k(s)\|_{L^2(B_\rho(x_0))}^2 ds \\ &= -2 \int_t^{t_0} \langle \zeta_m(s) \partial_s \zeta_m(s), |(u-k)^+(s)|^2 \rangle_{B_\rho(x_0)} ds + 2 \int_t^{t_0} \langle \zeta_m^2(s)(u-k)^+(s), f(s, u(s), \nabla u(s), v(s)) \rangle_{B_\rho(x_0)} ds \\ &\quad - 2 \int_t^{t_0} \langle \partial_j(\zeta_m^2(s)(u-k)^+(s)), a^{ij}(s) \partial_i u(s) + \sigma^{jr}(s) v^r(s) + g^j(s, u(s), \nabla u(s), v(s)) \rangle_{B_\rho(x_0)} ds \\ &\quad - 2 \int_t^{t_0} \langle \zeta_m^2(s)(u-k)^+(s), v^{r,k}(s) \rangle_{B_\rho(x_0)} dW_s^r + 2 \int_t^{t_0} \int_{B_\rho(x_0)} (u-k)^+(s, x) \zeta_m^2 \mu(ds, dx), \end{aligned}$$

where  $v^{r,k} := v^r 1_{\{u > k\}}$ .

Thus, by letting  $m \rightarrow \infty$  and the dominated convergence theorem, we can get

$$\|(u-k)^+(t)\zeta(t)\|_{L^2(B_\rho(x_0))}^2 + \int_t^{t_0} \|\zeta(s)v^k(s)\|_{L^2(B_\rho(x_0))}^2 ds$$

$$\begin{aligned}
&= -2 \int_t^{t_0} \langle \zeta(s) \partial_s \zeta(s), |(u-k)^+(s)|^2 \rangle_{B_\rho(x_0)} ds + 2 \int_t^{t_0} \langle \zeta^2(s) (u-k)^+(s), f(s, u(s), \nabla u(s), v(s)) \rangle_{B_\rho(x_0)} ds \\
&- 2 \int_t^{t_0} \langle \partial_j(\zeta^2(s) (u-k)^+(s)), a^{ij}(s) \partial_i u(s) + \sigma^{jr}(s) v^r(s) + g^j(s, u(s), \nabla u(s), v(s)) \rangle_{B_\rho(x_0)} ds \\
&- 2 \int_t^{t_0} \langle \zeta^2(s) (u-k)^+(s), v^{r,k}(s) \rangle_{B_\rho(x_0)} dW_s^r + 2 \int_t^{t_0} \int_{B_\rho(x_0)} (u-k)^+(s, x) \zeta^2 \mu(ds, dx).
\end{aligned}$$

Taking conditional expectation, we obtain

$$\begin{aligned}
&\|((u-k)\zeta)^+(t)\|_{L^2(B_\rho(x_0))}^2 + E \left[ \int_t^{t_0} \|\zeta(s) v^k(s)\|_{L^2(B_\rho(x_0))}^2 ds | \mathcal{F}_t \right] \\
&= -2E \left[ \int_t^{t_0} \langle \zeta(s) \partial_s \zeta(s), |(u-k)^+(s)|^2 \rangle_{B_\rho(x_0)} ds | \mathcal{F}_t \right] \\
&+ 2E \left[ \int_t^{t_0} \langle \zeta^2(s) (u-k)^+(s), f^k(s, (u(s)-k)^+, \nabla(u(s)-k)^+, v(s)) \rangle_{B_\rho(x_0)} ds | \mathcal{F}_t \right] \\
&- 2E \left[ \int_t^{t_0} \langle \partial_j(\zeta^2(s) (u-k)^+(s)), a^{ji}(s) \partial_i u(s) + \sigma^{jr}(s) v^r(s) \right. \\
&\quad \left. + g^{j,k}(s, (u(s)-k)^+, \nabla(u(s)-k)^+, v(s)) \rangle_{B_\rho(x_0)} ds | \mathcal{F}_t \right] \\
&+ 2E \left[ \int_t^{t_0} \int_{B_\rho(x_0)} (u-k)^+(s, x) \zeta^2 \mu(ds, dx) | \mathcal{F}_t \right], \tag{4.21}
\end{aligned}$$

where  $f^k(\cdot, \cdot, \cdot, X, Y, Z) := f(\cdot, \cdot, \cdot, X+k, Y, Z)$  and  $g^{j,k}(\cdot, \cdot, \cdot, X, Y, Z) := g^j(\cdot, \cdot, \cdot, X+k, Y, Z)$ . As  $k \geq k_0^+$ , the last term on the right hand side of (4.21) vanishes. Hence, starting from (4.21), we derive the desired result in a similar way to [15, Proposition 5.6].  $\square$

Given  $Q_{2\rho} := [t_0 - 4\rho^2, t_0] \times B_{2\rho}(x_0) \subset Q$  with  $\rho \in (0, 1)$ , let  $k_0^\pm = \text{esssup}_{\Omega \times Q_\rho} \hat{\xi}^\pm$ . Lemma 4.9 shows that  $u \in BSPDG^\pm(a_0^\pm, b_0, k_0^\pm, \eta; \rho, Q_\rho)$  with  $\eta = p$ .  $a_0^\pm = A_p(f_0^\pm, g_0; Q_\rho)$  and  $b_0$  given therein. On the other hand, in view of the local boundedness of weak solutions for BSPDEs ([15, Proposition 5.6 and Theorem 5.8]), we have

$$k_0^\pm \leq C \left\{ \rho^{-\frac{n+2}{2}} \|\hat{\xi}^\pm\|_{0,2;Q_{2\rho}} + A_p(\hat{f}^\pm, \hat{g}; Q_{2\rho}) \rho^{1-\frac{2+n}{p}} \right\}$$

with  $C$  depending on  $\lambda, \kappa, \varrho, \Lambda, n$  and  $p$ . Hence, further by Lemmas 4.8 and 4.9, we obtain finally the local behavior of weak solutions to the RBSPDE (1.1).

**Theorem 4.10.** *Let assumptions  $(\mathcal{A}_1)$ – $(\mathcal{A}_4)$  hold. Let  $(u, v, \mu)$  be a weak solution to the RBSPDE (1.1). Given  $Q_{2\rho} := [t_0 - 4\rho^2, t_0] \times B_{2\rho}(x_0) \subset Q$  with  $\rho \in (0, 1)$ , we have*

$$\begin{aligned}
\text{esssup}_{(\omega, s, x) \in \Omega \times Q_{\frac{\rho}{2}}} u^\pm &\leq C \left\{ \rho^{-\frac{n+2}{2}} (\|u^\pm\|_{0,2;Q_\rho} + \|\hat{\xi}^\pm\|_{0,2;Q_{2\rho}}) \right. \\
&\quad \left. + \left( A_p(f_0^\pm, g_0; Q_\rho) + A_p(\hat{f}^\pm, \hat{g}; Q_{2\rho}) \right) \rho^{1-\frac{2+n}{p}} \right\},
\end{aligned}$$

where  $C$  is a positive constant depending on  $\lambda, \kappa, \beta, \varrho, \Lambda, L, n$  and  $p$ .

## A Auxiliary lemmas and Itô's Formulas

This subsection states some useful lemmas and Itô formulas, which have been frequently used. The first lemma and corollary are from [4].

**Lemma A.1.** Let  $\{a_k, k \in \mathbb{N}\}$  be a sequence of nonnegative numbers satisfying

$$a_{k+1} \leq C_0 b^k a_k^{1+\delta},$$

where  $b > 1$ ,  $\delta > 0$  and  $C_0$  is a positive constant. Then, if  $a_0 \leq \theta_0 := C_0^{-\frac{1}{\delta}} b^{-\frac{1}{\delta^2}}$ , we have  $\lim_{k \rightarrow \infty} a_k = 0$ .

**Corollary A.2.** Let  $\phi : [r_0, \infty] \rightarrow \mathbb{R}^+$  be a nonnegative and decreasing function. Assume there exist constants  $C_1 > 0$ ,  $\alpha > 0$  and  $\varsigma > 1$  such that for any  $r_0 < r < l$ ,

$$\phi(l) \leq \frac{C_1}{(l-r)^\alpha} \phi(r)^\varsigma.$$

Then for any  $d$  satisfying

$$d \geq C_1^{\frac{1}{\alpha}} |\phi(r_0)|^{\frac{\varsigma-1}{\alpha}} 2^{\frac{\varsigma}{\varsigma-1}},$$

we have  $\phi(r_0 + d) = 0$ .

The following embedding lemma is from [15].

**Lemma A.3.** If for each  $t \in [0, T]$ ,  $u \in \mathcal{V}_2(\mathcal{O}_t)$ , then we have

$$\|u\|_{0, \frac{2(n+2)}{n}; \mathcal{O}_t} \leq C \|\nabla u\|_{0, 2; \mathcal{O}_t}^{\frac{n}{n+2}} \operatorname{esssup}_{(\omega, s) \in \Omega \times [t, T]} \|u(\omega, s)\|_{\frac{n}{n+2}}^{\frac{n}{n+2}} \leq C \|u\|_{\mathcal{V}_2(\mathcal{O}_t)},$$

where  $C$  only depends on  $n$ .

Now, we are going to present the Itô formulas, which have been frequently used in the main text. We assume that  $\Phi$  is a function that satisfies the following properties:

- (1)  $\Phi \in \mathcal{C}(\mathbb{R}^+ \times \mathbb{R}^n \times \mathbb{R} \rightarrow \mathbb{R})$  and  $\partial_t \Phi(t, x, u)$ ,  $\Phi'(t, x, u)$ ,  $\Phi''(t, x, u)$  and  $\partial_j \Phi'(t, x, u)$ ,  $j = 1, 2, \dots, n$  exist and are continuous;
- (2)  $\Phi'(t, x, 0) = 0$  for any  $(t, x) \in \mathbb{R}^+ \times \mathbb{R}^n$ ;
- (3)  $\sup_{t \in \mathbb{R}^+, x \in \mathbb{R}^n} |\partial_j \Phi'(t, x, u)| \leq C|u|$ ,  $j = 1, 2, \dots, n$ ;
- (4)

$$\sup_{t \in \mathbb{R}^+, x \in \mathbb{R}^n, u \in \mathbb{R} \setminus \{0\}} \left\{ |\Phi''(t, x, u)| + \frac{1}{|u|^2} |\partial_t \Phi(t, x, u) - \partial_t \Phi(t, x, 0)| \right\} < \infty,$$

where  $\partial_j \Phi(t, x, u) = \partial_{x_j} \Phi(t, x, u)$ ,  $\Phi'(t, x, u) = \partial_u \Phi(t, x, u)$  and  $\Phi''(t, x, u) = \partial_u^2 \Phi(t, x, u)$ .

Suppose that the following BSPDE

$$\begin{cases} -du(t, x) = [\partial_j(a^{ij} \partial_i u(t, x) + \sigma^{jr} v^r(t, x)) + \bar{f}(t, x) + \nabla \cdot \bar{g}(t, x)] dt \\ \quad + \mu(dt, x) - v^r(t, x) dW_t^r, \quad (t, x) \in Q, \\ u(T, x) = G(x), \quad x \in \mathcal{O}, \end{cases} \quad (\text{A.1})$$

holds in the weak sense where  $(u, v) \in \mathcal{V}_2(Q) \times \mathcal{M}^{0,2}(Q)$ ,  $\mu$  is a stochastic regular measure,  $\bar{f}$ ,  $\bar{g}$  and  $G$  satisfy  $(\mathcal{A}_3)$ ,  $a$  and  $\sigma$  satisfy  $(\mathcal{A}_2)$ .

When  $\Phi$  is independent of  $x$ , i.e.,  $\Phi(t, x, u) = \Phi(t, u)$ , the first Itô formula is from [16, Theorem 3.10].

**Proposition A.4.** Let BSPDE (A.1) hold in the weak sense with  $u|_{\partial\mathcal{O}} = 0$ . Then there holds almost surely that

$$\begin{aligned} & \int_{\mathcal{O}} \Phi(t, u(t, x)) dx + \frac{1}{2} \int_t^T \langle \Phi''(s, u(s)), |v(s)|^2 \rangle ds \\ &= \int_{\mathcal{O}} \Phi(T, G(x)) dx - \int_t^T \int_{\mathcal{O}} \partial_s \Phi(s, u(s, x)) dx ds + \int_t^T \langle \Phi'(s, u(s)), \bar{f}(s) \rangle ds \\ & \quad - \int_t^T \langle \Phi''(s, u(s)) \partial_j u(s), a^{ij}(s) \partial_i u(s) + \sigma^{jr}(s) v^r(s) + \bar{g}^j(s) \rangle ds \\ & \quad + \int_{[t, T] \times \mathcal{O}} \Phi'(s, u(s, x)) \mu(ds, dx) - \int_t^T \langle \Phi'(s, u(s)), v^r(s) \rangle dW_s^r. \end{aligned}$$

The following Itô formula extends the preceding one to the positive parts of the weak solutions to BSPDEs.

**Theorem A.5.** Let BSPDE (A.1) hold in the weak sense but with  $u^+|_{\partial\mathcal{O}} = 0$ . Then there holds almost surely that

$$\begin{aligned} & \int_{\mathcal{O}} \Phi(t, x, u^+(t, x)) dx + \frac{1}{2} \int_t^T \langle \Phi''(s, u^+(s)), |v^u(s)|^2 \rangle ds \\ &= \int_{\mathcal{O}} \Phi(T, x, G^+(x)) dx - \int_t^T \int_{\mathcal{O}} \partial_s \Phi(s, x, u^+(s, x)) dx ds \\ & \quad + \int_t^T \langle \Phi'(s, u^+(s)), \bar{f}^u(s) \rangle ds + \int_t^T \int_{\mathcal{O}} \Phi'(s, x, u^+(s, x)) \mu(ds dx) \\ & \quad - \int_t^T \langle \Phi''(s, u^+(s)) \partial_j u^+(s) + \partial_j \Phi'(s, u^+(s)), a^{ij}(s) \partial_i u^+(s) + \sigma^{jr}(s) v^{r,u}(s) + \bar{g}^{j,u}(s) \rangle ds \\ & \quad - \int_t^T \langle \Phi'(s, u^+(s)), v^{r,u}(s) \rangle dW_s^r, \end{aligned} \tag{A.2}$$

where

$$v^{r,u} = 1_{\{u>0\}} v^r, \quad \bar{f}^u = 1_{\{u>0\}} \bar{f}, \quad \bar{g}^{j,u} = 1_{\{u>0\}} \bar{g}^j.$$

*Proof.* Note that in general we cannot get  $u|_{\partial\mathcal{O}} = 0$  from  $u^+|_{\partial\mathcal{O}} = 0$ , so Proposition A.4 is not applicable. Instead, we shall apply an approximation scheme similar to that for [16, Theorem 3.10]

Let  $\tilde{u}$  be the stochastic regular parabolic potential (see next subsection for the definition) associated with  $\mu$ . Now define

$$\begin{cases} -d\hat{u}(t, x) = (-\Delta \hat{u}(t, x) + \bar{f}(t, x) + \nabla \cdot \hat{g}(t, x)) dt - v^r(t, x) dW_t^r, \\ (t, x) \in Q, \\ \hat{u}(0, x) = u(0, x), \quad x \in \mathcal{O}, \end{cases}$$

where  $\hat{g}^j(t, x) = \partial_j u(t, x) + a^{ij} \partial_i u(t, x) + \sigma^{jr} v^r(t, x) + \bar{g}^j(t, x)$ . Then,  $u = \hat{u} - \tilde{u}$  and the zero Dirichlet conditions of  $u^+$  and  $\tilde{u}$  imply  $\hat{u}^+|_{\partial\mathcal{O}} = 0$ . By [16, Proposition 3.9(i)]  $u$  is almost surely quasi-continuous. So, the integral w.r.t.  $\mu$  in (A.2) is well defined. We can also check that all the other terms in (A.2) are well defined.

Thus, by Proposition 3.9(iv) and Remark 3.7 in [16], there exist  $f^n \in \mathcal{L}^2([0, T]; (H^{-1})^+(\mathcal{O}))$ ,  $\tilde{v}^n \in \mathcal{L}^2([0, T]; (L^2(\mathcal{O}))^m)$ ,  $\tilde{u}^n \in \mathcal{U}(-\infty, f_1^n, g_1^n, G_1^n)$  and  $\phi^n \in \mathcal{U}(-\infty, f_2^n, g_2^n, G_2^n)$ , for some  $f_i^n \in \mathcal{L}^2([0, T]; L^2(\mathcal{O}))$ ,  $g_i^n \in \mathcal{L}^2([0, T]; (L^2(\mathcal{O}))^n)$ ,  $G_i^n \in \mathcal{L}^2(\mathcal{O})$ ,  $i = 1, 2$ , such that  $\phi^n \downarrow 0$  as  $n \rightarrow \infty$ ,  $dt \times dx \times d\mathbb{P}$  a.e.,  $\lim_{n \rightarrow \infty} \sum_{i=1}^m E \int_0^T \|\tilde{v}^{n,i}(t)\|^2 dt = 0$ ,  $\lim_{n \rightarrow \infty} \|\tilde{u}^n - \tilde{u}\|_{\mathcal{L}^2(\mathcal{K})} = 0$ ,  $\lim_{n \rightarrow \infty} (\|f_2^n + \nabla \cdot g_2^n\|_{\mathcal{L}^2([0, T]; H^{-1}(\mathcal{O}))} +$

$\|G_2^n\|_{\mathcal{L}^2(\mathcal{O})} = 0$ ,  $|\tilde{u}^n - \tilde{u}| \leq \phi^n \, dt \times dx \times d\mathbb{P}$  a.e., with  $\tilde{u}^n$  satisfying the SPDE

$$\begin{cases} d\tilde{u}^n(t, x) = [\Delta\tilde{u}^n(t, x) + f^n(t, x)] dt + \tilde{v}^n(t, x) dW_t, & (t, x) \in Q \\ \tilde{u}^n(0, x) = 0, & x \in \mathcal{O}, \\ \tilde{u}^n|_{\partial\mathcal{O}} = 0. \end{cases}$$

Define  $u^n := \hat{u} - \tilde{u}^n$ . Then

$$du^n(t, x) = -(-\Delta u^n(t, x) + \bar{f}(t, x) + f^n(t, x) + \nabla \cdot \hat{g}(t, x)) dt + (v^r(t, x) - \tilde{v}^{n,r}(t, x)) dW_t^r.$$

Moreover,  $|(u^n)^+ - u^+| \leq \phi^n \, dt \times dx \times d\mathbb{P}$  a.e.. The zero Dirichlet conditions of  $\tilde{u}^n$  and  $\hat{u}^+$  imply  $(u^n)^+|_{\partial\mathcal{O}} = 0$ . By [15, Lemma 3.5], we have almost surely

$$\begin{aligned} & \int_{\mathcal{O}} \Phi(t, x, (u^n)^+(t, x)) dx + \frac{1}{2} \int_t^T \langle \Phi''(s, (u^n)^+(s)), |(v(s) - \tilde{v}^n(s))1_{\{u^n > 0\}}|^2 \rangle ds \\ &= \int_{\mathcal{O}} \Phi(T, x, (u^n)^+(T, x)) dx - \int_t^T \int_{\mathcal{O}} \partial_s \Phi(s, x, (u^n)^+(s, x)) dx ds \\ & \quad + \int_t^T \langle \Phi'(s, (u^n)^+(s)), \bar{f}(s)1_{\{u^n > 0\}} \rangle ds + \int_t^T \langle \Phi'(s, (u^n)^+(s)), f^n(s)1_{\{u^n > 0\}} \rangle_{1, -1} ds \\ & \quad - \int_t^T \langle \Phi''(s, (u^n)^+(s)) \partial_j (u^n)^+(s) + \partial_j \Phi'(s, (u^n)^+(s)), -\partial_j (u^n)^+(s) + \hat{g}^j(s)1_{\{u^n > 0\}} \rangle ds \\ & \quad - \int_t^T \langle \Phi'(s, (u^n)^+(s)), (v^r(s) - \tilde{v}^{n,r}(s))1_{\{u^n > 0\}} \rangle dW_s^r, \quad \forall t \in [0, T]. \end{aligned} \tag{A.3}$$

By [16, Corollary 3.5], there exists  $\bar{u} \in \mathcal{L}^2(\mathcal{P})$  such that

$$|\hat{u}| + \phi^1 \leq \bar{u}, \quad dt \times dx \times d\mathbb{P} \text{ a.e..} \tag{A.4}$$

By (A.4) and the properties (2) and (4) of  $\Phi$ , there holds  $dt \times dx \times d\mathbb{P}$  a.e. that

$$\begin{aligned} |\Phi'(t, x, (u^n)^+(t, x))| &= |\Phi'(t, x, (u^n)^+(t, x)) - \Phi'(t, x, 0)| \\ &\leq C|u^n(t, x)| \\ &= C|\hat{u}(t, x) - \tilde{u}^n(t, x)| \\ &\leq C|\hat{u}(t, x)| + C|\tilde{u}(t, x)| + C|\tilde{u}(t, x) - \tilde{u}^n(t, x)| \\ &\leq C|\hat{u}(t, x)| + C|\tilde{u}(t, x)| + C\phi^n(t, x) \\ &\leq C(|\tilde{u}(t, x)| + \bar{u}(t, x)). \end{aligned} \tag{A.5}$$

By property (4) of  $\Phi$ , there holds  $dt \times dx \times d\mathbb{P}$  a.e. that

$$\begin{aligned} |\Phi'(t, x, (u^n)^+(t, x)) - \Phi'(t, x, u^+(t, x))| &\leq C|(u^n)^+(t, x) - u^+(t, x)| \\ &\leq C|\tilde{u}^n(t, x) - \tilde{u}(t, x)| \\ &\leq C\phi^n(t, x). \end{aligned} \tag{A.6}$$

(A.5), (A.6) and [16, Proposition 3.9(ii)] yield that

$$\begin{aligned} & \lim_{n \rightarrow \infty} \int_t^T \langle \Phi'(s, (u^n)^+(s)), f^n(s)1_{\{u^n > 0\}} \rangle_{1, -1} ds \\ &= \int_{\mathcal{O}_t} \Phi'(s, x, u^+(s, x)) \mu(ds dx) \text{ a.s..} \end{aligned}$$

Moreover,

$$E \sup_{0 \leq t \leq T} \left| \int_t^T \langle \Phi'(s, (u^n)^+(s)), (v^r(s) - \tilde{v}^{n,r}(s))1_{\{u^n > 0\}} \rangle dW_s^r - \int_t^T \langle \Phi'(s, u^+(s)), v^r(s)1_{\{u > 0\}} \rangle dW_s^r \right|$$

$$\begin{aligned}
&\leq CE \left( \int_0^T |\langle \Phi'(s, (u^n)^+(s)), (v^r(s) - \check{v}^{n,r}(s)) 1_{\{u^n > 0\}} \rangle - \langle \Phi'(s, u^+(s)), v^r(s) 1_{\{u > 0\}} \rangle|^2 ds \right)^{\frac{1}{2}} \\
&\leq CE \left( \int_0^T |\langle \Phi'(s, (u^n)^+(s)), v(s) 1_{\{u^n > 0\}} \rangle - \langle \Phi'(s, u^+(s)), v(s) 1_{\{u > 0\}} \rangle|^2 ds \right)^{\frac{1}{2}} \\
&\quad + CE \left( \int_0^T |\langle \Phi'(s, (u^n)^+(s)) - \Phi'(s, 0), \check{v}^n(s) 1_{\{u^n > 0\}} \rangle|^2 ds \right)^{\frac{1}{2}} \\
&\leq C (E \text{esssup}_{0 \leq t \leq T} \|(u^n)^+(t) - u^+(t)\|^2)^{\frac{1}{2}} \left( E \int_0^T \|v(t)\|^2 dt \right)^{\frac{1}{2}} \\
&\quad + C (E \text{esssup}_{0 \leq t \leq T} \|(u^n)^+(t)\|^2)^{\frac{1}{2}} \left( E \int_0^T \|v(t)(1_{\{u^n > 0\}} - 1_{\{u > 0\}})\|^2 dt \right)^{\frac{1}{2}} \\
&\quad + C (E \text{esssup}_{0 \leq t \leq T} \|u^n(t)\|^2)^{\frac{1}{2}} \left( E \int_0^T \|\check{v}^n(t)\|^2 dt \right)^{\frac{1}{2}} \\
&\leq C (E \|u^n - u\|_{\mathcal{K}}^2)^{\frac{1}{2}} \left( E \int_0^T \|v(t)\|^2 dt \right)^{\frac{1}{2}} \\
&\quad + C (E \|u^n\|_{\mathcal{K}}^2)^{\frac{1}{2}} \left( E \int_0^T \|v(t)(1_{\{u^n > 0\}} - 1_{\{u > 0\}})\|^2 dt \right)^{\frac{1}{2}} \\
&\quad + C (E \text{esssup}_{0 \leq t \leq T} \|u^n(t)\|^2)^{\frac{1}{2}} \left( E \int_0^T \|\check{v}^n(t)\|^2 dt \right)^{\frac{1}{2}} \\
&\rightarrow 0.
\end{aligned}$$

By the properties of  $\Phi$  and the fact that  $|(u^n)^+ - u^+| \leq \phi^n dt \times dx \times d\mathbb{P}$  a.e., the convergence of other terms can be treated analogously. Finally by letting  $n \rightarrow \infty$ , we obtain almost surely that

$$\begin{aligned}
&\int_{\mathcal{O}} \Phi(t, x, u^+(t, x)) dx + \frac{1}{2} \int_t^T \langle \Phi''(s, u^+(s)), |(v(s) 1_{\{u > 0\}})|^2 \rangle ds \\
&= \int_{\mathcal{O}} \Phi(T, x, u^+(T, x)) dx - \int_t^T \int_{\mathcal{O}} \partial_s \Phi(s, x, u^+(s, x)) dx ds \\
&\quad + \int_t^T \langle \Phi'(s, u^+(s)), \bar{f}(s) 1_{\{u > 0\}} \rangle ds + \int_t^T \int_{\mathcal{O}} \Phi'(s, x, u^+(s, x)) \mu(ds dx) \\
&\quad - \int_t^T \langle \Phi''(s, u^+(s)) \partial_j u^+(s) + \partial_j \Phi'(s, u^+(s)), -\partial_j u^+(s) + \hat{g}^j(s) 1_{\{u^n > 0\}} \rangle ds \\
&\quad - \int_t^T \langle \Phi'(s, u^+(s)), v^r(s) 1_{\{u > 0\}} \rangle dW_s^r, \quad \forall t \in [0, T].
\end{aligned}$$

□

## B Some definitions associated with stochastic regular measures

In general the random measure  $\mu$  in (1.1) can be a local time, which is not absolutely continuous w.r.t. Lebesgue measure. Hence, the Skorokhod condition  $\int_Q (u - \xi) \mu(dt, dx) = 0$  might not make sense. To give a precise meaning to the Skorokhod condition, the theory of parabolic potential and capacity introduced

by [12, 13] was generalized by [16] to a backward stochastic framework. This subsection recalls the notion of quasi continuity and stochastic regular measure, which are repeatedly used in the main text and in the proof of Theorem A.5. Moreover, spaces used in the proof of Theorem A.5 are also presented.

First some spaces are introduced. Denote by  $H_0^1(\mathcal{O})$  the first order Sobolev space vanishing on the boundary  $\partial\mathcal{O}$  equipped with the norm  $\|v\|_1^2 := \|v\|^2 + \|\nabla v\|^2$  and by  $H^{-1}(\mathcal{O})$  the dual space of  $H_0^1(\mathcal{O})$ . The dual pair between  $H_0^1(\mathcal{O})$  and  $H^{-1}(\mathcal{O})$  is denoted by  $\langle \cdot, \cdot \rangle_{1,-1}$ . Define  $(H^{-1})^+(\mathcal{O}) = \{v \in H^{-1}(\mathcal{O}) : \langle \varphi, v \rangle_{1,-1} \geq 0, \text{ for each } \varphi \in H_0^1(\mathcal{O}) \text{ and } \varphi \geq 0\}$ .

For a Hilbert space  $V$ , denote by  $\mathcal{L}^2([0, T]; V)$  the set of all  $L^2([0, T]; V)$  valued  $(\mathcal{F}_t)$  adapted process  $u$  with the norm defined as  $\|u\|_{\mathcal{L}^2([0, T]; V)} := \left( E\|u\|_{L^2([0, T]; V)}^2 \right)^{\frac{1}{2}} < \infty$ . Denote by  $\mathcal{L}^2(\mathcal{O})$  the set of all  $L^2(\mathcal{O})$  valued  $(\mathcal{F}_t)$  adapted process  $u$  with the norm  $\|u\|_{\mathcal{L}^2(\mathcal{O})} := (E\|u\|^2)^{\frac{1}{2}} < \infty$ .

Denote  $\mathcal{K} := L^\infty([0, T]; L^2(\mathcal{O})) \cap L^2([0, T]; H_0^1(\mathcal{O}))$ , equipped with the norm

$$\|v\|_{\mathcal{K}} := \left( \|v\|_{L^\infty([0, T]; L^2(\mathcal{O}))}^2 + \|v\|_{L^2([0, T]; H_0^1(\mathcal{O}))}^2 \right)^{\frac{1}{2}}.$$

Set  $\mathcal{W} = \{v \in L^2(0, T; H_0^1) : \partial_t v \in L^2(0, T; H^{-1})\}$  endowed with the norm

$$\|v\|_{\mathcal{W}} = \left( \|v\|_{L^2(0, T; H_0^1)}^2 + \|\partial_t v\|_{L^2(0, T; H^{-1})}^2 \right)^{\frac{1}{2}},$$

where  $H^{-1}$  is the dual space of  $H_0^1$ . Furthermore, we set

$$\mathcal{W}_T = \{v \in \mathcal{W} : v(T) = 0\}, \quad \mathcal{W}^+ = \{v \in \mathcal{W} : v \geq 0\}, \quad \mathcal{W}_T^+ = \mathcal{W}_T \cap \mathcal{W}^+.$$

**Definition B.1.** We denote by  $\mathcal{P}$  the set of parabolic potentials, which is the class of  $v \in \mathcal{K}$  such that

$$\int_0^T -\langle \partial_t \varphi(t), v(t) \rangle dt + \int_0^T \langle \partial_i \varphi(t), \partial_i v(t) \rangle dt \geq 0, \quad \forall \varphi \in \mathcal{W}_T^+.$$

Denote by  $\mathcal{C}(Q)$  the class of continuously differentiable functions in  $Q$  with compact support. By the Hahn-Banach theorem and because  $\mathcal{C}(Q) \cap \mathcal{W}_T$  is dense in  $\mathcal{C}(Q)$ , parabolic potentials can be represented by associated Radon measures. This leads to the following proposition, due to Pierre [13].

**Proposition B.2.** Let  $v \in \mathcal{P}$ . Then there exists a unique Radon measure on  $[0, T) \times \mathcal{O}$ , denoted by  $\mu^v$ , such that

$$\forall \varphi \in \mathcal{W}_T \cap \mathcal{C}(Q), \quad \int_0^T -\langle \partial_t \varphi(t), v(t) \rangle + \int_0^T \langle \partial_i \varphi(t), \partial_i v(t) \rangle dt = \int_0^T \int_{\mathcal{O}} \varphi(t, x) \mu^v(dt, dx)$$

**Definition B.3.** For any open set  $A \subset [0, T) \times \mathcal{O}$ , the parabolic capacity of  $A$  is defined as

$$\text{cap}(A) = \inf \{ \|\varphi\|_{\mathcal{W}}^2 : \varphi \in \mathcal{W}^+, \varphi \geq 1 \text{ a.e. on } A \}.$$

For any Borel set  $B \subset [0, T) \times \mathcal{O}$ , its parabolic capacity is defined as

$$\text{cap}(B) = \inf \{ \text{cap}(A) : A \supset B, A \text{ is open} \}.$$

**Definition B.4.** A real valued function  $\phi$  on  $[0, T) \times \mathcal{O}$  is said to be quasi-continuous, if there exists a sequence of non-increasing open sets  $A_n \subset [0, T) \times \mathcal{O}$  such that

- (1)  $\phi$  is continuous on the complement of each  $A_n$ ;
- (2)  $\lim_{n \rightarrow \infty} \text{cap}(A_n) = 0$ .



Denote by  $\mathcal{P}_0$  the class of  $v \in \mathcal{P}$  such that  $v$  is quasi-continuous and  $v(0) = 0$  in  $L^2$ . Each element  $v \in \mathcal{P}_0$  is called a regular potential and the associated Radon measure in Definition B.2 is called a regular measure. Furthermore, let  $\mathcal{L}^0(\mathcal{K})$  be the class of the measurable maps from  $(\Omega, \mathcal{F}_T)$  to  $\mathcal{K}$ , such that each element  $v \in \mathcal{L}^0(\mathcal{K})$  is an  $L^2$  valued adapted process.  $\mathcal{L}^0(\mathcal{P})$  and  $\mathcal{L}^0(\mathcal{P}_0)$  are similarly defined as  $\mathcal{L}^0(\mathcal{K})$ . Moreover, set

$$\mathcal{L}^2(\mathcal{K}) := L^2(\Omega, \mathcal{F}_T; \mathcal{K}) \cap \mathcal{L}^0(\mathcal{K})$$

endowed with the norm

$$\|v\|_{\mathcal{L}^2(\mathcal{K})} = (E\|v\|_{\mathcal{K}}^2)^{1/2}.$$

The stochastic parabolic potential is defined as

$$\mathcal{L}^2(\mathcal{P}) := \mathcal{L}^2(\mathcal{K}) \cap \mathcal{L}^0(\mathcal{P}),$$

endowed with the norm

$$\|u\|_{\mathcal{L}^2(\mathcal{P})} = \|u\|_{\mathcal{L}^2(\mathcal{K})}.$$

In addition, we define the stochastic regular parabolic potential as

$$\mathcal{L}^2(\mathcal{P}_0) := \mathcal{L}^2(\mathcal{P}) \cap \mathcal{L}^0(\mathcal{P}_0),$$

and the associated random Radon measure is called a stochastic regular measure.

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